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LECTURES ON POLYNOMIAL EQUATIONS: MAX NOETHER'S FUNDAMENTAL THEOREM, THE JACOBI FORMULA AND BÉZOUT'S THEOREM

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In memory of Jacek Chądzyński

Streszczenie. Using some commutative algebra we prove Max Noether's Theorem, the Jacobi Formula and Bézout's Theorem for systems of polynomial equations defining transversal hypersurfaces without common points at infinity.

The classical theorems on polynomial equations: Max Noether's Fundamental Theorem, The Jacobi Formula and Bézout's Theorem were presented in nineteenth-century literature (see for example [La] and [Ne]) for polynomial equations with indeterminate coefficients. In this article we give the present-day version of these theorems. To prove Max Noether's Fundamental Theorem which is basic for our approach we use Hilbert's Nullstellensatz and the Cohen-Macauley property of parameters. An elementary proof of the Cohen-Macauley property is given in [Pł].

1. Introduction

Let K be an algebraically closed field (of arbitrary characteristic). For any polynomial $P = P(X) \in K[X]$ in n variables $X = (X_1, \dots, X_n)$ we denote by $\deg P$ the total degree of P and by P^+ the principal part of P, i.e. the sum of all monomials of degree $\deg P$ appearing in P. By convention $\deg 0 = -\infty$, $0^+ = 0$.

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Definition 1. *Let $F_i \in K[X]$, $1 \leqslant i \leqslant n$ be nonconstant polynomials in n variables $X = (X_1, \ldots, X_n)$. The system of polynomial equations $F_1(X) = \cdots = F_n(X) = 0$ is* general *if the following conditions hold*

(1) *the system of polynomial equations $F_1(X) = \cdots = F_n(X) = 0$ has no solutions at infinity i.e. the system of homogeneous equations $F_1^+(X) = \cdots = F_n^+(X) = 0$ has in K^n only the zero-solution $X = 0$;*
(2) *all solutions in K^n of the system $F_1(X) = \cdots = F_n(X) = 0$ are simple i.e. the jacobian* $\det\left(\frac{\partial F_i}{\partial X_j}\right)$ *does not vanish on the solutions of this system.*

Let us consider some examples:

(1) The system of linear equations $a_{i1}X_1 + \cdots + a_{in}X_n - b_i = 0$, $1 \leqslant i \leqslant n$ is general if and only if $\det(a_{ij}) \neq 0$.
(2) If $F_i = X_i^{d_i} + c_{i1}X_i^{d_i-1} + \cdots + c_{id_i} \in K[X_i]$, $1 \leqslant i \leqslant n$, are one-variable polynomials of degree $d_i > 0$ with simple roots then the system $F_1(X_1) = \cdots = F_n(X_n) = 0$ is general.
(3) Let $s_i(X)$, $1 \leqslant i \leqslant n$ be symmetric polynomials defined by identity
$$(T - X_1)\cdots(T - X_n) = T^n + s_1(X)T^{n-1} + \cdots + s_n(X)$$
i.e.
$$s_1(X) = -(X_1 + \cdots + X_n), \cdots, \quad s_n(X) = (-1)^n X_1 \cdots X_n.$$
Let $D(s_1, \ldots, s_n)$ be the discriminant of the polynomial $T^n + s_1T^{n-1} + \cdots + s_n$ with general coefficients $s_1, \ldots, s_n$. Recall that
$$D(s_1(X), \ldots, s_n(X)) = \left(\det\left(\frac{\partial s_i(X)}{\partial X_j}\right)\right)^2 = \prod_{i=1, i>j}^{n} (x_i - x_j)^2$$
(see pages 150-151 of [Pe]).
It is easy to see that the system of polynomial equations $s_1(X) - a_1 = \cdots = s_n(X) - a_n = 0$, where $a_i \in K$, is general if and only if $D(a_1, \ldots, a_n) \neq 0$.

In the sequel we put $F = (F_1, \ldots, F_n) \in K[X]^n$, $\operatorname{Jac} F = \det\left(\frac{\partial F_i(X)}{\partial X_j}\right)$ and $V(F) = \{x = (x_1, \ldots, x_n) \in K^n \ : \ F_1(x) = \cdots = F_n(x) = 0\}$. The system of polynomial equations $F_1(X) = \cdots = F_n(X) = 0$ will be denoted $F = 0$.

Now we may formulate the three classical theorems mentioned in the title of these lectures.

Theorem 1 (Max Noether's Fundamental Theorem)**.** *Let $F = 0$ be a general system of polynomial equations. If a polynomial G vanishes on the set $V(F)$ then there exists polynomials $A_1, \ldots, A_n \in K[X]$ such that*
$$G = \sum_{i=1}^{n} A_i F_i \quad \text{and} \ \deg A_i F_i \leqslant \deg G \quad \text{for } i \in \{1, \ldots, n\}.$$

We will give the proof of Theorem 1 in Section 3 of these notes. Note that with the notations of Theorem 1 we have $\deg G = \max_{i=1}^{n}(\deg A_i F_i)$ since the inequality $\deg G \leqslant \max_{i=1}^{n}(\deg A_i F_i)$ is obvious. The following property is an immediate consequence of Max Noether's Theorem.

Corollary 1. *The solutions of the general system of polynomial equations $F_1(X) = \cdots = F_n(X) = 0$ do not lie on a hypersurface of degree strictly less than $\min_{i=1}^{n}(\deg F_i)$. Moreover the system $F_1(X) = \cdots = F_n(X) = 0$ has at least one solution in K^n.*

Proof. If the solutions of the system $F_1(X) = \cdots = F_n(X) = 0$ lie on the hypersurface $G(X) = 0$ then $\deg G = \max_{i=1}^{n}(\deg A_i F_i) \geqslant \min_{i=1}^{n}(\deg F_i)$. This proves the first assertion. To check the second assertion suppose that the system $F_1(X) = \cdots = F_n(X) = 0$ has no solutions in K^n. Taking $G = 1$ we get $\deg G \geqslant \min_{i=1}^{n}(\deg F_i) > 0$ by the first part of the corollary. Contradiction.

Using Max Noether's Fundamental Theorem we prove in Section 4

Theorem 2 (The Jacobi Formula)**.** *Let $F = 0$ be a general system of polynomial equations. Then the set $V = V(F)$ is finite and for every polynomial $H \in K[X]$ of degree* $\deg H < \sum_{i=1}^{n}(\deg F_i - 1)$ *one has*

$$\sum_{x \in V(F)} \frac{H(x)}{\operatorname{Jac} F(x)} = 0.$$

Note that if $n = 1$ then the Jacobi Formula follows easily from the Lagrange Interpolation Theorem: let $F(X) = (X - x_1) \cdots (X - x_d) \in K[X]$ be a univariate polynomial of degree $d > 1$ such that $x_i \neq x_j$ for $i \neq j$. Then

$$H(X) = \sum_{i=1}^{d} \frac{H(x_i)}{F'(x_i)} (X - x_1) \cdots (\widehat{X - x_i}) \cdots (X - x_d)$$

provided that $H(X)$ is a polynomial of degree strictly less than d.

The assumption on the degree of H cannot be weakened. If $\operatorname{char} K = 0$ then $H = \operatorname{Jac} F$ is of degree $\sum_{i=1}^{n}(\deg F_i - 1)$ and $\sum_{x \in V(F)} \frac{H(x)}{\operatorname{Jac} F(x)} = \sharp V(F) \neq 0$.

Corollary 2 (The Cayley-Bacharach Theorem)**.** *If a polynomial H of degree strictly less than $\sum_{i=1}^{n}(\deg F_i - 1)$ vanishes on all points of $V = V(F)$ but one then it necessarily vanishes on V.*

The oldest result on general systems of polynomial equations is due to Étienne Bézout (Théorie générale des équations algébriques, Paris, 1770).

Theorem 3 (Bézout's Theorem)**.** *Let $F = 0$ be a general system of polynomial equations. Then it has exactly $\prod_{i=1}^{n} \deg F_i$ solutions.*

We give the proof of Theorem 3 in Section 3. To prove Béout's Theorem we will use Max Noether's Fundamental Theorem and the Poincaré series (see Section 5).

2. Homogeneous systems of parameters

Let $\phi = (\phi_1, \dots, \phi_n)$ be a sequence of homogeneous polynomials $\phi_i \in K[X]$, $X = (X_1, \dots, X_n)$. Using Hilbert's Nullstellensatz we check

Lemma 1. *Let K be an algebraically closed field. Then the following conditions are equivalent:*

1. *the system of homogeneous equations $\phi_1(X) = \dots = \phi_n(X) = 0$ has in K^n only the zero-solution $X = 0$.*
2. *there is an integer $N > 0$ such that all monomials $X_1^{\alpha_1} \cdots X_n^{\alpha_n}$, $\alpha_1 + \dots + \alpha_n = N$ belong to the ideal $I(\phi) = (\phi_1, \dots, \phi_n)K[X]$ generated by $\phi_1, \dots, \phi_n$ in $K[X]$.*

Now let K be an arbitrary field.

Definition 2. *The sequence of homogeneous forms $\phi = (\phi_1, \dots, \phi_n) \in K[X]^n$ is a homogeneous system of parameters (h.s.o.p.) if the ideal generated by $\phi_1, \dots, \phi_n$ in $K[X]$ contains all monomials of sufficiently high degree i.e. if it satisfies the second condition of the above lemma.*

The following result on h.s.o.p. is basic for us. For the proof see [St] (page 37, *The Cohen-Macauley property*).

Theorem 4. *If $\phi = (\phi_1, \dots, \phi_n) \in K[X]^n$ is a h.s.o.p. then for every k, $0 < k < n$ and for every homogeneous polynomial ψ such that $\psi\phi_{k+1} \in (\phi_1, \dots, \phi_k)K[X]$ we have $\psi \in (\phi_1, \dots, \phi_k)K[X]$.*

3. Proof of Max Noether's Fundamental Theorem

Let $F_1, \dots, F_n \in K[X]$ be polynomials (we do not assume that the system $F_1(X) = \dots = F_n(X) = 0$ is general!) in n variables $X = (X_1, \dots, X_n)$ with coefficients in an algebraically closed field K. Let $G \in K[X]$. We say that the sequence $G, F_1, \dots, F_n$ satisfies Noether's conditions at $x \in K^n$ if there exists a polynomial $D_x = D_x(X) \in K[X]$ such that $D_x(x) \neq 0$ and $D_x G$ is in the ideal $(F_1, \dots, F_n)K[X]$.

Lemma 2. *Let $G, F_1, \dots, F_n \in K[X]$ be polynomials such that for every $x \in K^n$ the sequence $G, F_1, \dots, F_n$ satisfies Noether's conditions at x. Then $G \in (F_1, \dots, F_n)K[X]$.*

Proof. The system of polynomial equations $D_x(X) = 0$, $x \in K^n$ has no solutions in K^n. Therefore by Hilbert's Nullstellensatz there exists a family of polynomials $M_x(X)$, $x \in K^n$ such that $\sharp\{x \in K^n \ : \ M_x(X) \neq 0\} < +\infty$ and $\sum_{x \in K^n} M_x D_x = 1$ in $K[X]$. Then we get $G = \left(\sum_{x \in K^n} M_x D_x \right) G = \sum_{x \in K^n} M_x (D_x G) \in (F_1, \dots, F_n)K[X]$.

Remark 1. *If $x \notin V(F_1, \dots, F_n)$ then for any polynomial G the sequence $G, F_1, \dots, F_n$ satisfies Noether's conditions at x. It suffices to take $D_x = F_i$ where F_i is such that $F_i(x) \neq 0$.*

Lemma 3. *Let $F_1, \dots, F_n \in K[X]$ be polynomials such that $F_1(x) = \dots = F_n(x) = 0$ and $\det\left(\frac{\partial F_i}{\partial X_j}(x)\right) \neq 0$ at a point $x = (x_1, \dots, x_n) \in K^n$. Then there is a polynomial $D_x(X) \in K[X]$ such that $(X_i - x_i)D_x \in (F_1, \dots, F_n)K[X]$ for $i \in \{1, \dots, n\}$ and $D_x(x) \neq 0$.*

Proof. Write $F_i(X) = (X_1 - x_1)D_{i1}(X) + \dots + (X_n - x_n)D_{in}(X)$ in $K[X]$ for $i \in \{1, \dots, n\}$. Differentiating and putting $X = x$ we get $D_{ij}(x) = \frac{\partial F_i}{\partial X_j}(x)$. Let $D_x(X) := \det(D_{ij}(X))$. Then $D_x(x) \neq 0$ and by Cramer's Rule $(X_i - x_i)D_x(X) \in (F_1, \dots, F_n)K[X]$.

Proposition 1. *Let $F_1, \dots, F_n \in K[X]$ be polynomials such that for every $x \in V(F_1, \dots, F_n)$ one has $\det\left(\frac{\partial F_i}{\partial X_j}(x)\right) \neq 0$. Let $G \in K[X]$ be a polynomial such that $G(x) = 0$ for all $x \in V(F_1, \dots, F_n)$. Then $G \in (F_1, \dots, F_n)K[X]$.*

Proof. Let $x = (x_1, \dots, x_n) \in K^n$. If $x \in V(F_1, \dots, F_n)$ then $G(X) = \sum_{i=1}^{n} (X_i - x_i)G_i(X)$. By Lemma 3 there is a polynomial $D_x(X) \in K[X]$ such that $(X_i - x_i)D_x(X) \in (F_1, \dots, F_n)K[X]$. Thus $D_x G \in (F_1, \dots, F_n)K[X]$. By Lemma 2 and Remark 1 we get $G \in (F_1, \dots, F_n)K[X]$.

What remains to be proved in Noether's Theorem is the bound on the degrees.

Proposition 2. *Let $F_1, \dots, F_n \in K[X]$ be nonconstant polynomials such that the homogeneous forms $F_i^+ \in K[X]$, $i \in \{1, \dots, n\}$, form a h.s.o.p. Then for every $G \in (F_1, \dots, F_n)K[X]$ there exists $A_1, \dots, A_n \in K[X]$ such that $G = \sum_{i=1}^{n} A_i F_i$ and $\deg(A_i F_i) \leqslant \deg(G)$ for $i \in \{1, \dots, n\}$.*

Proof. Let X_0 be a new variable and let $\tilde{G}(X_0, X)$, $\tilde{F}_i(X_0, X)$, $i \in \{1, \dots, n\}$, be the homogenization of $G(X)$ and $F_i(X)$ for $i \in \{1, \dots, n\}$. Recall that $\tilde{G}(X_0, X) = X_0^{\deg G} G\left(\frac{X_1}{X_0}, \dots, \frac{X_n}{X_0}\right)$. Since $G \in (F_1, \dots, F_n)K[X]$ we get $X_0^N \tilde{G} \in (\tilde{F}_1, \dots, \tilde{F}_n)K[X_0, X]$ for an integer $N > 0$. It is easy to see that $X_0^N, \tilde{F}_1, \dots, \tilde{F}_n$ form a h.s.o.p. in $K[X_0, X]$. By Theorem 4 X_0^N is not a zero-divisor mod $(\tilde{F}_1, \dots, \tilde{F}_n)$ and we may write $\tilde{G} = \sum_{i=1}^{n} \psi_i \tilde{F}_i$ where ψ_i are homogeneous polynomials such that $\psi_i \tilde{F}_i$ is either 0 or of degree $\deg \tilde{G}$. Let $A_i(X) = \psi_i(1, X)$ for $i \in \{1, \dots, n\}$. Putting $X_0 = 1$ in the identity $\tilde{G} = \sum_{i=1}^{n} \psi_i \tilde{F}_i$ we get $G = \sum_{i=1}^{n} A_i F_i$ and $\deg(A_i F_i) \leqslant \deg G$ for $i \in \{1, \dots, n\}$.

Remark 2. *With the assumptions of Proposition 2 one has $\max_{i=1}^{n}(\deg A_i F_i) = \deg G$ and $G^+ = \sum_{i \in I} A_i^+ F_i^+$ where $I = \{i : \deg(A_i F_i) = \deg(G)\}$. In particular $G^+ \in (F_1^+, \dots, F_n^+)$.*

Proof of Max Noether's Fundamental Theorem. Max Noether's Theorem follows immediately from Proposition 1 and Proposition 2.

4. Proof of the Jacobi formula

Lemma 4. *Let $F = (F_1, \dots, F_n) \in K[X]^n$ be polynomials with coefficients in a field K. Then the set $W = \{x \in K^n : F(x) = 0 \text{ and } \operatorname{Jac} F(x) \neq 0\}$ is finite.*

Proof. By Lemma 3 for every $x \in W$ there is a polynomial $D_x = D_x(X)$ such that $D_x(x) \neq 0$ and

$$(X_i - x_i) D_x \in (F_1, \dots, F_n) \text{ for } i = 1, \dots, n.$$

Let us put $U_x = \{\tilde{x} \in K^n : D_x(\tilde{x}) \neq 0\}$ for every $x \in W$. Then $U_x \subseteq K^n$ is a Zariski open subset of K^n and $W \cap U_x = \{x\}$. Since $K[X]$ is a noetherian ring there exists a finite sequence $x_1, \dots, x_s \in W$ such that $\bigcup_{x \in W} U_x = \bigcup_{i=1}^{s} U_{x_i}$. Obviously $W = \{x_1, \dots, x_s\}$.

Now, let $F=(F_1,\dots,F_n)\in K[X]^n$ be a sequence of polynomials such that the set $V=V(F)$ is finite. If $R,S\in K[X]$ and $S(x)\neq 0$ for all $x\in V$ then we define the *trace* of $\dfrac{R}{S}$ with respect to F by putting $\operatorname{Tr}_F\left(\dfrac{R}{S}\right):=\sum_{x\in V}\dfrac{R(x)}{S(x)}$.

If the system of polynomial equations $F=0$ has only simple solutions then $\operatorname{Tr}_F\left(\dfrac{H}{\operatorname{Jac}F}\right)=\sum_{x\in V}\dfrac{H(x)}{\operatorname{Jac}F(x)}$ is well-defined.

Lemma 5. *Let $F=(F_1,\dots,F_n)\in K[X]^n$ and $G=(G_1,\dots,G_n)\in K[X]^n$ be such that the systems of polynomial equations $F=0$ and $G=0$ have only simple zeroes. Suppose that $G_i=\sum_{j=1}^n A_{ij}F_j$ in $K[X]$. Let $A=\det(A_{ij})$. Then $\operatorname{Tr}_F\left(\dfrac{H}{\operatorname{Jac}F}\right)=\operatorname{Tr}_G\left(\dfrac{AH}{\operatorname{Jac}G}\right)$.*

Proof. Differentiating the identities

$$G_i=\sum_{j=1}^n A_{ij}F_j \tag{1}$$

we get

$$\operatorname{Jac}G\equiv A\operatorname{Jac}F \pmod{(F_1,\dots,F_n)K[X]}. \tag{2}$$

From (1) and (2) we get that for all $x\in K^n$, $F(x)=0$ if and only if $G(x)=0$ and $A(x)\neq 0$. Indeed, if $F(x)=0$ then $G(x)=0$ by (1) and $\operatorname{Jac}G(x)=A(x)\operatorname{Jac}F(x)$ by (2). Thus $\operatorname{Jac}G(x)\neq 0$ by the hypothesis that all the zeroes of the system $G=0$ are simple, consequently we get $A(x)\neq 0$.

On the other hand suppose that $G(x)=0$ and $A(x)\neq 0$. Then from (1) we get $0=\sum_{j=1}^n A_{ij}(x)F_j(x)$ for $i\in\{1,\dots,n\}$ and $F_j(x)=0$ by Cramer's Rule. Summing up we have $V(F)=V(G)\backslash V(A)$ and $\operatorname{Jac}G=A\operatorname{Jac}F$ on $V(F)$.

Now, we get

$$\operatorname{Tr}_F\left(\frac{H}{\operatorname{Jac}F}\right)=\sum_{x\in V(F)}\frac{H(x)}{\operatorname{Jac}F(x)}=$$
$$=\sum_{x\in V(G)\backslash V(A)}\frac{A(x)H(x)}{\operatorname{Jac}G(x)}=\sum_{x\in V(G)}\frac{A(x)H(x)}{\operatorname{Jac}G(x)}=\operatorname{Tr}_G\left(\frac{AH}{\operatorname{Jac}G}\right).$$

Lemma 6. *If $G=(G_1,\dots,G_n)\in K[X]^n$ where $G_i=G_i(X_i)\in K[X_i]$, $i\in\{1,\dots,n\}$, are nonconstant polynomials with simple zeroes then for every polynomial $H\in K[X]$, $\deg H<\sum_{i=1}^n(\deg G_i-1)$ one has $\operatorname{Tr}_G\left(\dfrac{H}{\operatorname{Jac}G}\right)=0$.*

Proof. By linearity of the trace we may assume that $H = X_1^{a_1} \cdots X_n^{a_n}$. It is easy to see that $\operatorname{Tr}_G\left(\dfrac{H}{\operatorname{Jac} G}\right) = \operatorname{Tr}_{G_1}\left(\dfrac{X_1^{a_1}}{G_1'}\right) \cdots \operatorname{Tr}_{G_n}\left(\dfrac{X_n^{a_n}}{G_n'}\right)$. If $\deg H = \sum_{i=1}^n a_i < \sum_{i=1}^n (\deg G_i - 1)$ then $a_i < \deg G_i - 1$ for some $i \in \{1, \dots, n\}$ and $\operatorname{Tr}_{G_i}\left(\dfrac{X_i^{a_i}}{G_i'}\right) = 0$. Consequently $\operatorname{Tr}_G\left(\dfrac{H}{\operatorname{Jac} G}\right) = 0$ and we are done.

Proof of the Jacobi Formula. Let $F = 0$ be a general system of polynomial equations. Then the set $V = V(F)$ is finite by Lemma 4 (and non-empty by Corollary 1). Let $\Pi_i : K^n \longrightarrow K$ be the projection given by $\Pi_i(x_i, \dots, x_n) = x_i$ and put $G_i(X_i) = \prod_{x_i \in V_i} (X_i - x_i) \in K[X_i]$ where $V_i = \Pi_i(V(F))$. Then $G_i(X_i)$ is a polynomial with simple zeroes vanishing on V. By Max Noether's Fundamental Theorem we may write $G_i = A_{i1}F_1 + \cdots + A_{in}F_n \in K[X]$ with $\deg(A_{ij}F_j) \leqslant \deg G_i$ for $i \in \{1, \dots, n\}$. Let $A = \det(A_{ij})$. For any permutation $(j_1, \dots, j_n)$ of $(1, \dots, n)$ we get $\deg(\pm A_{1j_1} \cdots A_{nj_n}) \leqslant (\deg G_1 - \deg F_{j_1}) + \cdots + (\deg G_n - \deg F_{j_n}) = \sum_{i=1}^n (\deg G_i - \deg F_i)$ and consequently $\deg A \leqslant \sum_{i=1}^n (\deg G_i - \deg F_i)$.

Let $H \in K[X]$ be a polynomial such that $\deg H < \sum_{i=1}^n (\deg F_i - 1)$. Therefore $\deg(AH) < \sum_{i=1}^n (\deg G_i - \deg F_i) + \sum_{i=1}^n (\deg F_i - 1) = \sum_{i=1}^n (\deg G_i - 1)$. Let $G = (G_1, \dots, G_n)$. By Lemma 5 and Lemma 6 we get $\operatorname{Tr}_F\left(\dfrac{H}{\operatorname{Jac} F}\right) = \operatorname{Tr}_G\left(\dfrac{AH}{\operatorname{Jac} G}\right) = 0$.

5. Poincaré series

Let K be an arbitrary field (not necessarily algebraically closed).

Let $\phi_1, \dots, \phi_n \in K[X]$, $X = (X_1, \dots, X_n)$ be a sequence of homogeneous forms of degrees $d_1, \dots, d_n > 0$. For any integer $d \geqslant 0$ we denote by $K[X]_d$ the linear K-linear subspace of $K[X]$ generated by monomials $X_1^{\alpha_1} \cdots X_n^{\alpha_n}$, $\alpha_1 + \cdots + \alpha_n = d$. For any integer m, $1 \leqslant m \leqslant n$ we put $(\phi_1, \dots, \phi_m)_d$ the K-linear subspace of $K[X]_d$ consisted of the sums $\alpha_1\phi_1 + \cdots + \alpha_m\phi_m$ where α_i are homogeneous polynomials such that $\alpha_i\phi_i$ is either 0 or of degree d. We put, by convention, $(\phi_1, \dots, \phi_m)_d = (0)_d$ if $m = 0$.

Theorem 5. *Suppose that $\phi_1, \dots, \phi_n$ is a sequence of homogeneous parameters in $K[X]$. Then for any integer m, $0 \leqslant m \leqslant n$ we have*

$$\sum_{d \geqslant 0} (\dim_K K[X]_d/(\phi_1, \dots, \phi_m)_d)\, T^d = \frac{\prod_{i=1}^m (1 - T^{d_i})}{(1 - T)^n}.$$

Remark 3. *The formal power series which appears on the left side of the above identity is named the* Poincaré series *of the graded algebra* $K[X]/(\phi_1,\dots,\phi_m) \simeq \bigoplus_{d\geqslant 0} K[X]_d/(\phi_1,\dots,\phi_m)_d$.

To prove Theorem 5 we need two lemmas.

Lemma 7. $\sum_{d\geqslant 0}(\dim_K K[X]_d)\,T^d = \dfrac{1}{(1-T)^n}$.

Proof. Let $T_1,\dots,T_n$ be new variables. Then

$$\left(\sum_{\alpha_1\geqslant 0} T_1^{\alpha_1}\right)\cdots\left(\sum_{\alpha_n\geqslant 0} T_n^{\alpha_n}\right) = \sum_{(\alpha_1,\dots,\alpha_n)\in\mathbf{N}^n} T_1^{\alpha_1}\cdots T_n^{\alpha_n}.$$

Let T be a variable. Substituting $T_1=\dots=T_n=T$ we get

$$\left(\sum_{\alpha\geqslant 0} T^{\alpha}\right)^n = \sum_{(\alpha_1,\dots,\alpha_n)\in\mathbf{N}^n} T^{\alpha_1+\dots+\alpha_n} =$$
$$= \sum_{d\geqslant 0}\left(\sum_{\alpha_1+\dots+\alpha_n=d} 1\right)T^d = \sum_{d\geqslant 0}(\dim_K K[X]_d)T^d$$

and the Lemma follows since $\sum_{\alpha\geqslant 0} T^{\alpha} = \dfrac{1}{1-T}$ in $\mathbf{Z}[T]$.

Lemma 8.

(1) $\dim_K K[X]_d/(\phi_1,\dots,\phi_m)_d = \dim_K K[X]_d/(\phi_1,\dots,\phi_{m-1})_d$ for $d<d_m$.
(2) $\dim_K K[X]_d/(\phi_1,\dots,\phi_m)_d = \dim_K K[X]_d/(\phi_1,\dots,\phi_{m-1})_d -$
$- \dim_K K[X]_{d-d_m}/(\phi_1,\dots,\phi_{m-1})_{d-d_m}$ for $d\geqslant d_m$.

Proof. Property *1.* is obvious since $(\phi_1,\dots,\phi_m)_d = (\phi_1,\dots,\phi_{m-1})_d$ for $d<d_m$. Let U be a K-linear space of finite dimension. Then for any subspaces $W,V\subseteq U$ such that $W\subseteq V$ we have $\dim_K U/W = \dim_K U/V + \dim_K V/W$. Taking $U = K[X]_d$, $V=(\phi_1,\dots,\phi_m)_d$ and $V=(\phi_1,\dots,\phi_{m-1})_d$ we get

$$\text{(3)}\quad \begin{aligned}\dim_K K[X]_d/(\phi_1,\dots,\phi_{m-1})_d &= \dim_K K[X]_d/(\phi_1,\dots,\phi_m)_d\\ &+ \dim_K(\phi_1,\dots,\phi_m)_d/(\phi_1,\dots,\phi_{m-1})_d.\end{aligned}$$

By Theorem 4 ϕ_m is not a zero-divisor mod $(\phi_1,\dots,\phi_{m-1})$. Consequently the mapping $A\longrightarrow A\phi_m$ where $A\in K[X]_{d-d_m}$ induces an isomorphism of spaces $(\phi_1,\dots,\phi_m)_d/(\phi_1,\dots,\phi_{m-1})_d$ and $K[X]_{d-d_m}/(\phi_1,\dots,\phi_{m-1})_{d-d_m}$ and we get

$$\text{(4)}\quad \dim_K(\phi_1,\dots,\phi_m)_d/(\phi_1,\dots,\phi_{m-1})_d = \dim_K K[X]_{d-d_m}/(\phi_1,\dots,\phi_{m-1})_{d-d_m}.$$

From (3) and (4) we obtain Property *2.* of Lemma.

Now we can give

Proof of Theorem 5.

If $m = 0$ then the formula follows from Lemma 7. Suppose that $m > 0$ and that Theorem 5 holds for $m - 1$. So we have

$$\sum_{d\geqslant 0} (\dim_K K[X]_d/(\phi_1,\ldots,\phi_{m-1})_d)\, T^d = \frac{(1-T^{d_1})\cdots(1-T^{d_m-1})}{(1-T)^n}.$$

Using Lemma 8 we get

$$\begin{aligned}
&\sum_{d\geqslant 0} (\dim_K K[X]_d/(\phi_1,\ldots,\phi_m)_d)\, T^d = \\
&\quad = \sum_{d\geqslant 0} (\dim_K K[X]_d/(\phi_1,\ldots,\phi_{m-1})_d)\, T^d - \\
&\qquad\qquad - \sum_{d\geqslant d_m} (\dim_K K[X]_{d-d_m}/(\phi_1,\ldots,\phi_{m-1})_{d-d_m})\, T^d = \\
&\quad = \frac{(1-T^{d_1})\cdots(1-T^{d_{m-1}})}{(1-T)^n} - \frac{(1-T^{d_1})\cdots(1-T^{d_{m-1}})}{(1-T)^n} T^{d_m} = \\
&\quad = \frac{(1-T^{d_1})\cdots(1-T^{d_m})}{(1-T)^n}.
\end{aligned}$$

Corollary 3. *If $\phi_1,\ldots,\phi_n$ is a system of homogeneous parameters in $K[X]$ with $\deg\phi_i = d_i$, then*

$$\dim_K K[X]/(\phi_1,\ldots,\phi_n) = d_1\cdots d_n.$$

Proof. If $m = n$ then by Theorem 5 we get

$$\begin{aligned}
&\sum_{d\geqslant 0} (\dim_K (K[X]_d/(\phi_1,\ldots,\phi_n)_d)\, T^d = \\
&\qquad = (1+T+\cdots+T^{d_1-1})\cdots(1+T+\cdots+T^{d_n-1}).
\end{aligned} \tag{5}$$

Therefore $\dim_K (K[X]_d/(\phi_1,\ldots,\phi_n)_d) = 0$ for $d > \sum_{i=1}^{n}(d_i - 1)$. Substituting $T = 1$ in (5) we get

$$\sum_{d\geqslant 0} \dim_K K[X]_d/(\phi_1,\ldots,\phi_n)_d = d_1\cdots d_n.$$

It suffices to observe that $K[X]/(\phi_1,\ldots,\phi_n)$ and $\oplus_{d\geqslant 0} K[X]_d/(\phi_1,\ldots,\phi_n)_d$ are K-isomorphic.

Let $d_1, \ldots, d_n > 0$ be a sequence of positive integers. For any $d \geqslant 0$ we put

$$(6)\quad \nu_d(d_1, \ldots, d_n) = \sharp \left\{ (\alpha_1, \ldots, \alpha_n) : 0 \leqslant \alpha_i < d_i \text{ for } i = 1, \ldots, n,\ \sum_{i=1}^{n} \alpha_i = d \right\}.$$

Lemma 9. *To abbreviate the notation we put $\nu_d = \nu_d(d_1, \ldots, d_n)$.*

(i) $(1 + T + \ldots T^{d_1 - 1}) \ldots (1 + T + \ldots T^{d_n - 1}) = \sum_{d \geqslant 0} \nu_d,$

(ii) $\sum_{d \geqslant 0} \nu_d = d_1 \ldots d_n,$

(iii) *Let* $\delta = \sum_{i=1}^{n} (d_i - 1)$. *Then* $\nu_d = \nu_{\delta - d}$ *for* $0 \leqslant d \leqslant \delta$.

Proof. Property (i) is obvious. Putting $T = 1$ we get (ii). The polynomial on the left side of (i) is recurrent, hence it follows (iii).

Proposition 3. $\dim_K K[X]_d / (\phi_1, \ldots, \phi_n)_d = \nu_d(d_1, \ldots, d_n)$.

Proof. Use formula (5) and Lemma 9 (i) .

6. Monomial bases

We keep the notation and assumptions of Section 5. In particular, K is an arbitrary field. Let $A = K[X]/I$ be an affine algebra of finite dimension $D = \dim_K A$. A monomial basis of A mod. the ideal I is a sequence of monomials $e_0, \ldots, e_{D-1} \in K[X]$ such that the images of $e_0, \ldots, e_{D-1}$ in A form a linear basis of A.

Proposition 4. *Let $F_1, \ldots, F_n \in K[X]$ be nonconstant polynomials such that the homogeneous forms $F_1^+, \ldots, F_n^+$ form h.s.o.p. Let $I(F) = (F_1, \ldots, F_n)$ and $I(F^+) = (F_1^+, \ldots, F_n^+)$. Then any monomial basis mod $I(F^+)$ is a monomial basis mod $I(F)$.*

Proof. Let $\epsilon_0, \epsilon_1, \ldots, \epsilon_{D-1}$ be a monomial basis. We will check that $\epsilon_0, \epsilon_1, \ldots, \epsilon_{D-1}$ is a linear basis mod $I(F)$. First, let us prove that $\epsilon_0, \epsilon_1, \ldots, \epsilon_{D-1}$ are linearly independent mod $I(F)$. Suppose that there is a non-zero sequence $c_0, \ldots, c_{D-1} \in K$ such that $c_0\epsilon_0 + \cdots + c_{D-1}\epsilon_{D-1} \equiv 0 \mod I(F)$. Let $I = \{i : c_i \neq 0\}$ and $I_0 = \{i \in I : \deg(\sum_j c_j \epsilon_j) = \deg \epsilon_i\}$. Then, by Remark 2 we get $\sum_{i \in I_0} c_i \epsilon_i \equiv 0 \pmod{I(F^+)}$ which contradicts the linear independence of ϵ_i mod $I(F^+)$.

To check that every polynomial G is a linear combination of ϵ_i mod $I(F)$ we use induction on $\deg G$. Let $N > 0$ be an integer and suppose that every polynomial of degree strictly less than N is a linear combination of ϵ_i mod $I(F)$. Let G be a polynomial of degree N. It suffices to check that G^+ is a linear combination of

$\epsilon_0, \dots, \epsilon_{D-1}$ mod $I(F)$. Since $\epsilon_0, \dots, \epsilon_{D-1}$ form a linear basis mod $I(F^+)$ we may write

$$G^+ = \phi_1 F_1^+ + \dots + \phi_n F_n^+ + \sum_i c_i \epsilon_i$$

where ϕ_i are homogeneous forms such that $\phi_i F_i^+$ is of degree $\deg G^+ = N$. Write $F_i = F_i^+ + R_i$, $1 \leqslant i \leqslant n$, where $\deg R_i < \deg F_i^+$. Then we get

$$G^+ = \phi_1(F_1 - R_1) + \dots + \phi_n(F_n - R_n) + \sum_i c_i \epsilon_i \equiv$$
$$\equiv \phi_1(-R_1) + \dots + \phi_n(-R_n) + \sum_i c_i \epsilon_i \bmod I(F)$$

where $\deg(-\phi_1 R_1 - \dots - \phi_n R_n) < N$ and we are done.

Theorem 6. *If $F_1, \dots, F_n$ are nonconstant polynomials, $d_1 = \deg F_1, \dots, d_n = \deg F_n$ such that the forms $F_1^+, \dots, F_n^+$ form a homogeneous system of parameters then*

$$\dim_K K[X]/I(F) = d_1 \dots d_n \, .$$

Proof. Proposition (4) implies that $\dim_K K[X]/I(F) = \dim_K K[X]/I(F^+)$. Use Corollary 3.

Theorem 7. *With the assumptions of Theorem 6 there exists a monomial basis mod the ideal $I(F)$ such that*

$$\sharp\{i : \deg e_i = d\} = \nu_d(d_1, \dots, d_n)$$

for any $d \geqslant 0$.

Proof. According to Proposition 4 it suffices to prove the theorem for ideal $I(F^+)$. Let $\epsilon_0, \epsilon_1, \dots, \epsilon_{D-1}$ be a monomial basis mod $I(F^+)$. Fix an integer $d \geqslant 0$. Since $K[X]/I(F^+) = \bigoplus K[X]_d/I(F^+)_d$ the images of $\epsilon_0, \epsilon_1, \dots, \epsilon_{D-1}$ of degree d form a basis of the space $K[X]_d/I(F^+)_d$ which is of dimension $\nu_d(d_1, \dots, d_n)$ by Proposition 3.

7. Proof of Bézout's Theorem

We keep the notations of Introduction. We consider a general system of polynomial equations $F = 0$ and its set of solutions $V(F)$. We know that $V(F)$ is non-empty (see Corollary 1) and finite (see Lemma 4). Let us denote $I(F)$ the ideal generated by polynomials $F_1, \dots, F_n$ in the ring $K[X]$. To prove Bézout's Theorem we need

Lemma 10.

$$\sharp V(F) = \dim_K K[X]/I(F).$$

Proof. Let us consider the K-algebra $K[V]$ of polynomial functions on the set $V = C(F)$. It is easy to see that the family $\{e_x : x \in V\}$ where $e_x(x) = 1$ and $e_x(x') = 0$ for $x' \in V \setminus \{x\}$ is a K-linear basis of $K[V]$. Thus $\dim_K K[V] = \sharp V$. On the other hand the K-linear homomorphism $\sigma : K[X] \to K[V]$ defined by $\sigma(P) = P_{|V}$, has by Proposition 1 the kernel $I(V)$. Thus $K[V]$ and $K[V]/I(F)$ are isomorphic and the lemma follows.

Proof of Theorem 3. By Lemma 10 and Theorem 6 we have

$$\sharp V(F) = \dim_K K[X]/I(F) = \prod_{i=1}^{n} \deg F_i.$$

The reader will find more about Bézout's Theorem in [LJ].

8. Application to real algebraic geometry

Let $F = (F_1, \ldots, F_n) \in \mathbb{R}[X]^n$ be nonconstant polynomials in n variables $X = (X_1, \ldots, X_n)$ of degrees $d_1, \ldots, d_n > 0$. Suppose that the system of polynomial equations $F = 0$ is general (see Definition 1). Let $V = V(F)$ be the set of all complex solutions of $F = 0$ and let $V_{\mathbb{R}} = V(F) \cap \mathbb{R}^n$. Let $J_F = \operatorname{Jac} F$. We define $\operatorname{ind} F = \sum_{a \in V_{\mathbb{R}}} \operatorname{sgn} J_F(a)$ (the index of vector field F). We define the Petrovskii number $\Pi(d_1, \ldots, d_n)$ by the formula

$$\Pi(d_1, \ldots, d_n) = \sharp \left\{ (\alpha_1, \ldots, \alpha_n) : 0 \leqslant \alpha_i < d_i, \sum_{i=1}^{n} \alpha_i = \frac{1}{2} \sum_{i=1}^{n} (d_i - 1) \right\}.$$

Clearly, if $\sum_{i=1}^{n} (d_i - 1)$ is an odd number then $\Pi(d_1, \ldots, d_n) = 0$. Note also that $\Pi(d_1, d_2) = \min\{d_1, d_2\}$ if $d_1 + d_2 \equiv 0 \pmod 2$.

The following theorem may be considered as a real counterpart of Bézout's theorem.

Theorem 8 (Petrovskii-Oleinik Inequality)**.** *With the notation and assumptions introduced above*

$$|\operatorname{ind} F| \leqslant \Pi(d_1, \ldots, d_n).$$

The inequality figuring in Theorem 8 was proved by Arnold [A] and called by him the Petrovskii-Oleinik inequality. Khovanskii [Kh] proved an inequality of this type for the index of polynomial vector field in the open set defined by an equation $P > 0$.

Proof of the Petrovskii-Oleinik inequality.

1. Preliminaries

Let $V \subset \mathbb{C}^n$ be a finite subset of $\mathbb{C}^n$ such that if $a = (a_1, \ldots, a_n) \in V$ then $\overline{a} = (\overline{a_1}, \ldots, \overline{a_n}) \in V$. Let $\mathbb{R}[V]$ be the set of all functions $f : V \to \mathbb{C}$ such that $\overline{f(a)} = f(\overline{a})$ for $a \in V$. Then $\mathbb{R}[V]$ is an algebra over $\mathbb{R}$, $\dim_{\mathbb{R}} \mathbb{R}[V] = \sharp V$. Let $\phi \in \mathbb{R}[V]$ be a fixed function which is nowhere 0. We consider the bilinear form B_ϕ on $\mathbb{R}[V]$ defined by

$$B_\phi(f, g) = \sum_{a \in V} \phi(a) f(a) g(a).$$

Lemma 11. *The quadratic form $Q_\phi(f) = B_\phi(f, f)$ takes real values and is non-degenerate. The signature $\sigma(Q_\phi)$ of Q_ϕ is equal to*

$$\sum_{a \in V \cap \mathbb{R}} \operatorname{sgn} \phi(a).$$

Proof of Lemma 11. Let $V = \{a_1, \ldots, a_r, b_1, \ldots, b_s, \overline{b_1}, \ldots, \overline{b_s}\}$ where $\overline{a_i} = a_i$ for $i = 1, \ldots, r$, $\overline{b_j} \neq b_j$ for $j = 1, \ldots, s$ are pairwise different. We have

$$Q_\phi(f) = \sum_{i=1}^{r} \phi(a_i) f(a_i)^2 + 2 \sum_{j=1}^{s} \operatorname{Re} \{\phi(b_j) f(b_j)\}.$$

Let $Q_i(f) = \phi(a_i) f(a_i)^2$ $(i = 1, \ldots, r)$ and $R_j(f) = \phi(b_j) f(b_j)^2$ $(j = 1, \ldots, s)$. Then $\operatorname{rank} Q_i = 1$, $\sigma(Q_i) = \operatorname{sgn} \phi(a_i)$, $\operatorname{rank} R_j = 2$, $\sigma(R_j) = 0$. The subspaces corresponding to linear forms $f \to f(a_i)$ and $f \to f(b_j)$ are orthogonal with respect to the form B_ϕ. Therefore

$$\begin{aligned}\operatorname{rank} Q_\phi &= \operatorname{rank} Q_1 + \cdots + \operatorname{rank} Q_r + \operatorname{rank} R_1 + \cdots + \operatorname{rank} R_s \\ &= r + 2s = \sharp V = \dim_{\mathbb{R}} \mathbb{R}[V]\end{aligned}$$

and

$$\sigma(Q_\phi) = \sigma(Q_1) + \cdots + \sigma(Q_r) + \sigma(R_1) + \cdots + \sigma(R_s) = \sum_{i=1}^{r} \operatorname{sgn} \phi(a_i).$$

Lemma 12. *Let N be any linear subspace of $\mathbb{R}[V]$ on which Q_ϕ is identically equal to zero. Then $\sigma(Q_\phi) \leqslant \dim_{\mathbb{R}} \mathbb{R}[V] - 2 \dim_{\mathbb{R}} N$.*

Proof. The lemma follows from Witt's theorem (see [L]), p. 592, Corollary 10.4).

Let V be the set of all complex solutions of the general system of real equations $F_1 = 0, \ldots, F_n = 0$ of degrees $d_1, \ldots, d_n > 0$. Note that $\dim_{\mathbb{R}} \mathbb{R}[V] = \sharp V = \dim_{\mathbb{C}} \mathbb{C}[V] = d_1 \cdots d_n$ by Bézout's theorem. For any polynomial $H \in \mathbb{R}[X_1, \ldots, X_n]$

we define a function $[H]$ of $\mathbb{R}[V]$ by putting $[H](a) = H(a)$ for $a \in V$. Let us consider the subspace of $\mathbb{R}[V]$:

$$N = \left\{ [H] \in \mathbb{R}[V] : \deg H < \frac{1}{2} \sum_{i=1}^{n} (d_i - 1) \right\}.$$

If $[H] \in N$ then $\deg H^2 < \sum_{i=1}^{n} (d_i - 1)$ and by the Jacobi formula

$$\sum_{a \in V} \frac{H(a)^2}{\operatorname{Jac} F(a)} = 0.$$

Let $\phi = \dfrac{1}{\operatorname{Jac} F}$. Then the subspace N is contained in the cone $Q_\phi^{-1}(0)$. By Lemma 12 we get

$$\begin{aligned} |\operatorname{ind} F| = \left| \sum_{a \in V} \operatorname{sgn} \operatorname{Jac} F(a) \right| = |\sigma(Q_\phi)| \leqslant \dim_{\mathbb{R}} \mathbb{R}[V] - 2 \dim_{\mathbb{R}} N \\ < d_1 \cdots d_n - 2 \dim_{\mathbb{R}} N. \end{aligned}$$

By Theorem 7 there exists a monomial basis $e_0, \ldots, e_n$ of $\mathbb{R}[V]$ such that

$$\sharp\{i : \deg e_i = d\} = \nu_d(d_1, \ldots, d_n) \quad \text{for } d \geqslant 0.$$

Let $\delta = \sum_{i=1}^{n} (d_i - 1)$. Then

$$\begin{aligned} \dim_{\mathbb{R}} N &= \textit{number of elements in monomial basis of degree } < \frac{1}{2}\delta \\ &= \textit{number of elements in monomial basis of degree } > \frac{1}{2}\delta \end{aligned}$$

by Lemma 9 (iii).

Therefore

$$\begin{aligned} 2 \dim_{\mathbb{R}} N &= \textit{number of elements in monomial basis of degree } \neq \frac{1}{2}\delta \\ &= d_1 \ldots d_n - \nu_{\frac{1}{2}\delta}(d_1, \ldots, d_n) \end{aligned}$$

and

$$|\operatorname{ind} F| = |\sigma(Q_\phi)| \leqslant \dim_{\mathbb{R}} \mathbb{R}[V] - 2 \dim_{\mathbb{R}} N = \nu_{\frac{1}{2}\delta}(d_1, \ldots, d_n) = \Pi(d_1, \ldots, d_n).$$

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