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EFFECTIVE PROOF OF GUSEĬN-ZADE THEOREM THAT BRANCHES MAY BE DEFORMED WITH JUMP ONE

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Abstract. Let $f \in \mathbb{C}\{X,Y\}$ be a reduced series which defines a singular branch $f = 0$ in a neighbourhood of zero in $\mathbb{C}^2$. Let $\mathbf{h} \geq 1$ be the number of characteristic exponets of a Puiseux root $y(X) \in \mathbb{C}\{X\}^*$ of the equation $f = 0$. For any $k \in \{1,\dots,\mathbf{h}\}$ we define the series $f_k \in \mathbb{C}\{X,Y\}$ generated by all terms of the series $y(X)$ with orders strictly smaller than the k-th characteristic exponent. We consider a deformation $F_t = f + tX^{\omega_0} f_1^{\omega_1} \dots f_{\mathbf{h}}^{\omega_{\mathbf{h}}}$ ($t \neq 0$, small) where $\omega_0, \omega_1, \dots, \omega_{\mathbf{h}}$ are nonnegative integers. Using a version of the Newton algorithm proposed by Cano we show how to choose exponents $\omega_0, \omega_1, \dots, \omega_{\mathbf{h}}$ to obtain the Milnor number of the deformation F_t smaller by one than the Milnor number of the branch f. We prove a version of Kouchnirenko theorem which is useful in computation the Milnor number.

1. Introduction

Let $f \in \mathbb{C}\{X,Y\}$ be a reduced series which defines an isolated singularity in the neighbourhood of $0 \in \mathbb{C}^2$ and let $F \in \mathbb{C}\{T,X,Y\}$ be a series such that $F(0,X,Y) = f(X,Y)$ and $F_t \in \mathbb{C}\{X,Y\}$ are isolated singularities for small $t \in \mathbb{C}$. The series F is called a deformation of the singularity of f. For any series $g,h \in \mathbb{C}\{X,Y\}$ the intersection multiplicity $(g,h)_0$ is defined as the $\mathbb{C}$ codimension of the ideal generated by g and h in $\mathbb{C}\{X,Y\}$. We consider the Milnor number $\mu(f) = (\partial f/\partial X, \partial f/\partial Y)_0$. At Arnold's seminar they asked what happened with the Milnor number of the singularity after deformation ([1], e.g. 1975–15, 1982–12). The semi-continuity of the Milnor number implies that $\mu(f) \geq \mu(F_t)$ (see: e.g. [9]). A basic notion that can be studied in this context is the minimal jump

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of the Milnor number $\mu(f) - \mu(F_t)$ where F_t runs over all deformations of singularity. In [10] Guseĭn-Zade showed that there exist reducible singularities which the minimal jump greater than one. Moreover, he proved that this jump equals one for branches. The proofs of the above mentioned results are not effective. The effective proof of the second result is the aim of this note. The effective proof of the first type result was obtained by Brzostowski and Krasiński in [3]. Many results concerning deformations of homogeneous singularities can be found in [4].

Bodin in [2] used the Kouchnirenko theorem [14] in order to obtain an effective construction of the deformation. We recall the Kouchnirenko theorem in dimension two. For any series $f = \sum c_{\alpha\beta} X^\alpha Y^\beta$ we consider its Newton diagram $\Delta(f)$ which is the convex hull of the union of the sets $(\alpha, \beta) + \mathbb{R}_+^2$ where (α, β) runs over all nonzero coefficients of f; $\mathbb{R}_+ = \{x : x \geq 0\}$. Assume that the Newton diagram has the vertex $(a, 0)$ on the horizontal axis and the vertex $(0, b)$ on the vertical axis. Note that if f is singular then $a, b \geq 2$. Let P denotes the area of the polygon bounded by the boundary of the diagram $\Delta(f)$ and by the coordinate axes. The Kouchnirenko theorem states that $\mu(f) \geq 2P - a - b + 1$.

In order to describe the equality case in the formula of Kouchnirenko we need the notion of nondegeneracy. We consider the Newton polygon $\mathcal{N}(f)$ which is the set of compact boundary faces (pairwise nonparallel) of the Newton diagram $\Delta(f)$. For any face (segment) $S \in \mathcal{N}(f)$ we define the initial form $\mathrm{in}(f, S)$ as the sum of all monomials $c_{\alpha\beta} X^\alpha Y^\beta$ of f such that $(\alpha, \beta) \in S$. We say that the series f is nondegenerate on S if the initial form $\mathrm{in}(f, S) \in \mathbb{C}[X, Y]$ has only single factors different from the powers of variables X or Y. We say that the series f is nondegenrate (in Kouchnirenko sense) if it is nondegenerate on every segment of the Newton polygon $\mathcal{N}(f)$. In the case of nondegeneracy we have the equality in the formula of Kouchnirenko. The opposite implication is true in dimention two (see e.g. [7]).

For any coprime integers p and q such that $p > q \geq 2$ let us consider a nondegenerate singularity $f = X^p + Y^q$. In the mentioned paper, Bodin proposed the deformation $F_t = X^p + Y^q + tX^{\tilde{\alpha}} Y^{\tilde{\beta}}$. Using the elementary number theory it is possible to choose $(\tilde{\alpha}, \tilde{\beta})$ below the segment joining $(0, q)$ and $(p, 0)$ such that $0 < \tilde{\alpha} < p$, $0 < \tilde{\beta} < q$ and the area of the triangle with vertices $(0, q)$, $(\tilde{\alpha}, \tilde{\beta})$, $(p, 0)$ equals $\frac{1}{2}$. By Kouchnirenko Theorem we get $\mu(F_t) = \mu(f) - 1$ for $t \neq 0$. This idea was developed by Michalska and Walewska in [21]. They showed for the considered singularity that every number from 1 to $r(q - r)$ can be the jump of the Milnor number of f where r is the remainder of division p by q.

The main result of our note (Theorem 1.1) may be treated as a generalization of the mentioned above observation of Bodin. Before presenting the result let us recall the ring of Puiseux sereis $\mathbb{C}\{X\}^* = \bigcup_{N \geq 1} \mathbb{C}\{X^{1/N}\}$ [23, 22, 20, 18]. For any positive integer number v_0 we consider a series $y \in \mathbb{C}\{X^{1/v_0}\}$. For nonzero y we can write

$$y = a_1 X^{v_1/v_0} + a_2 X^{v_2/v_0} + \dots\,, \quad a_1, a_2, \dots \neq 0\,, \tag{1}$$

$0 < v_1 < v_2 < \dots$ integers. We call $(v_0, v_1, v_2, \dots)$ a sequence associated with y. With $y = 0$ we associate the sequence (v_0). The elements of every two sequences associated with $y \in \mathbb{C}\{X\}^*$ are proportional. Therefore, there exists exactly one sequence associated with y for which the greatest common divisor of its elements equals 1. Let $\mathcal{G}(v_0)$ denotes the group of unity roots of degree v_0. For every $\tau \in \mathcal{G}(v_0)$ we define the action

$$\tau * y = a_1 \tau^{v_1} X^{v_1/v_0} + a_2 \tau^{v_2} X^{v_2/v_0} + \dots \, . \tag{2}$$

Let τ be a primitive root of $\mathcal{G}(v_0)$. The series $\tau^0 * y, \tau^1 * y, \dots, \tau^{v_0-1} * y$ are called the conjugations of y in $\mathbb{C}\{X^{1/v_0}\}$. The conjugation of the zero series equals itself. The number of different conjugations of y equals $N = v_0/\mathrm{GCD}(v_0, v_1, \dots)$ (see: e.g. [20]). We obtain them for $i = 0, 1, \dots, N-1$. The different conjugations form the so-called cycle of series y. The number N and the cycle depend only on the series y. We write $N = N(y)$ for the number of elements and cycle(y) for the cycle.

By using Newton-Puiseux theorem (see e.g. [23], [18]) we conclude that for every branch f coprime to X there exists a series $y \in \mathbb{C}\{X^{1/v_0}\}$ with N-elemental cycle $\{\tau^0 * y, \tau^1 * y, \dots, \tau^{N-1} * y\}$, $N = v_0/\mathrm{GCD}(v_0, v_1, \dots)$, $\tau \in \mathcal{G}(v_0)$ a primitive root, such that the equality

$$f(X,Y) = \prod_{i=0}^{N-1} (Y - \tau^i * y) \tag{3}$$

is satisfied up to a unit factor. An argument of Galois theory shows that fractional powers do not appear on the right side of (3) [22]. We can assume this unit factor to be one without loss of generality.

By definition, a characteristic exponent of the series $y \in \mathbb{C}\{X\}^*$ is an exponent which can appear as the order of difference between the series y and its conjugation (e.g. [20]). The exponent v_ℓ/v_0 ($\ell = 1, 2, \dots$) is characteristic if and only if

$$\mathrm{GCD}(v_0, \dots, v_{\ell-1}) > \mathrm{GCD}(v_0, \dots, v_\ell) \, . \tag{4}$$

The number $\mathbf{h} = \mathbf{h}(y)$ of characteristic exponents is less than or equal to $N(y) - 1$. Moreover, $\mathbf{h}(y) = 0 \Leftrightarrow N(y) = 1$ Let $\ell_1 < \dots < \ell_{\mathbf{h}}$ denote the characteristic positions and let $w^* = w^*(y) = \mathrm{GCD}(v_0, v_1, \dots)$. We define the Puiseux characteristic $(b_0, b_1, \dots, b_{\mathbf{h}})$ as $b_0 := v_0/w^*$, $b_1 := v_{\ell_1}/w^*$, ..., $b_{\mathbf{h}} := v_{\ell_{\mathbf{h}}}/w^*$, the first sequence of divisors $e_k := \mathrm{GCD}(b_0, b_1, \dots, b_k)$ ($k = 0, 1, \dots, \mathbf{h}$) and the second sequence of divisors $n_k = e_{k-1}/e_k$ ($k = 1, \dots, \mathbf{h}$). We put $N_0 := 1$ and $N_k := n_1 \dots n_k$ for $k = 1, \dots, \mathbf{h}$. We have $N_k = b_0/e_k$ for $k = 0, 1, \dots, \mathbf{h}$. Classical characteristics of branches are described in [25].

For every $k \in \{1, 2, \dots, \mathbf{h}\}$ we define the series y_k as the sum of all terms of y of order strictly less then b_k/b_0. The cycle of y_k has N_{k-1} elements. We put

$$f_k(X,Y) = \prod_{i=0}^{N_{k-1}-1} (Y - \tau^i * y_k) \in \mathbb{C}[X,Y] \tag{5}$$

where $\tau \in \mathcal{G}(v_0)$ is a primitive root. The following theorem is the main result of this paper.

Theorem 1.1. *Let $f \in \mathbb{C}\{X,Y\}$ be a singular branch and let $y \in \mathbb{C}\{X\}^*$ be a Puiseux root of the equation $f = 0$. Let $\mathbf{h} = \mathbf{h}(y)$ be the number of characteristic exponents ($\mathbf{h} \geq 1$) and let $f_1, \ldots, f_{\mathbf{h}}$ be the series generated from y by cutting below the characteristic exponents. Then there exist nonnegative integers $\omega_0, \omega_1, \ldots, \omega_{\mathbf{h}}$ such that the Milnor number of the deformation $F_t = f + tX^{\omega_0} f_1^{\omega_1} \ldots f_{\mathbf{h}}^{\omega_{\mathbf{h}}}$ ($t \neq 0$, small) equals $\mu(f) - 1$.*

In chapter 2 we present the Newton algorithm in version of Cano [5, 19]. In chapter 3 we present a variant of the Kouchnirenko theorem adopted to the Newton algorithm. In the last chapter of this note we prove Theorem 1.1.

2. The Newton Algorithm

Let us introduce some usefull notions. For any segment S of the Newton polygon we consider its *inclinaction* which is a rational number $|S|_{\mathbf{H}}/|S|_{\mathbf{V}}$ where $|S|_{\mathbf{H}}$ (resp. $|S|_{\mathbf{V}}$) is the lenght of the projection of S on the horizontal axis (resp. on the vertical axis). For a nonzero series $y \in \mathbb{C}\{X\}^*$ we define its initial form $\operatorname{in} y = aX^{\theta}$ ($a \neq 0$) as the term with the minimal order. By convention we put $\operatorname{in} 0 = 0$ and $\operatorname{ord} 0 = +\infty$. Let $f \in \mathbb{C}\{X,Y\}$ be a nonzero series and let $y \in \mathbb{C}\{X\}^*$ be a series of a positive order such that $\operatorname{in} y = aX^{\theta}$. Isaac Newton (in the letter to Odenburg) presented an observation that if y is a nonzero root of the series f (i.e. $f(X, y(X)) = 0$ in $\mathbb{C}\{X\}^*$) then there exists a segment S of the Newton polygon $\mathcal{N}(f)$ of inclination θ such that the initial form $\operatorname{in} y = aX^{\theta}$ is a nonzero root of the initial form $\operatorname{in}(f,S)$ in $\mathbb{C}\{X\}^*$. Therefore, the Newton polygon gives us the information about the orders of all nonzero solutions (of positive order). Moreover, we can read the number of such solutions from the shape of $\mathcal{N}(f)$. We denote by $\delta(f)$ the distance between the diagram $\Delta(f)$ and the horizontal axis. The zero solution $y = 0$ appears if and only if $\delta(f) > 0$.

The information of iniatial forms of solutions $y \in \mathbb{C}\{X\}^*$ of the equation $f = 0$ may be expressed by using systems (see: notion of symmetric power [24]). For elements $a_1, \ldots, a_p$ of a given set by the system $\mathcal{A} = \langle a_1, \ldots, a_p \rangle$ we mean the sequence $a_1, \ldots, a_p$ treated as unordered. We put $\deg \mathcal{A} = p$. Instead of

$$\langle \underbrace{a_1, \ldots, a_1}_{m_1 \text{ times}}, \ldots, \underbrace{a_p, \ldots, a_p}_{m_p \text{ times}} \rangle$$

we write $\langle a_1 : m_1, \ldots, a_p : m_p \rangle$. For $\mathcal{A} = \langle a_1, \ldots, a_p \rangle$ and $\mathcal{B} = \langle b_1, \ldots, b_q \rangle$ we have a natural addition $\mathcal{A} \oplus \mathcal{B} = \langle a_1, \ldots, a_p, b_1, \ldots, b_q \rangle$ with the neutral element $\langle\ \rangle$ (empty system). By convention $\langle a : 0 \rangle = \langle\ \rangle$.

Let $f \in \mathbb{C}\{X,Y\}$ be a series such that $p := (f, X)_0 = \operatorname{ord} f(0,Y) \geq 1$. Let us denote by $\operatorname{Zer} f$ the system $\langle y_1, \ldots, y_p \rangle$ of solutions of the equation $f = 0$ in $\mathbb{C}\{X\}^*$. For $S \in \mathcal{N}(f)$ let $\operatorname{in}(f,S)^{\circ}$ denotes the form $\operatorname{in}(f,S)$ divided by the maximal possible powers of variables X and Y.

Theorem 2.1. (Newton-Puiseux) *Then*

(i) $\langle \operatorname{ord} y_1, \dots, \operatorname{ord} y_p \rangle = \bigoplus_{S \in \mathcal{N}(f)} \langle |S|_{\mathbf{H}}/|S|_{\mathbf{V}} : |S|_{\mathbf{V}} \rangle \oplus \langle +\infty : \delta(f) \rangle$,

(ii) $\langle \operatorname{in} y_1, \dots, \operatorname{in} y_p \rangle = \bigoplus_{S \in \mathcal{N}(f)} \operatorname{Zer} \operatorname{in}(f, S)^\circ \oplus \langle 0 : \delta(f) \rangle$,

(iii) $p = |\mathcal{N}(f)| + \delta(f)$.

Now, let aX^θ be a nozero root of an intial form $\operatorname{in}(f, S)$, $S \in \mathcal{N}(f)$. By Isaac Newton observation aX^θ is the first term of a Puiseux solution of $f = 0$ in $\mathbb{C}\{X\}^*$. In order to find the second term Cano [5] consider the substitution

$$\tilde{f} = f(X, aX^\theta + Y)\ . \tag{6}$$

Observing the Newton diagram $\Delta f(X, aX^\theta + Y)$ he look for the boundary segments $S \in \mathcal{N}(\tilde{f})$ with the inclination stricly greater then θ. Then he choose the second term as a nonzero root of $\operatorname{in}(\tilde{f}, S)$. He continue the process to construct all nonzero terms of all nonzero solutions.

In order to deal with substitutions of the type (6) we apply the ring $\mathbb{C}\{X^*, Y\} = \sum_{N \geq 1} \mathbb{C}\{X^{1/N}, Y\}$ and we analogously define all neccessary notions. In comparison to the classical algorithm, Cano's approche allows to analyze every step of the algorithm in the same coordinate system. The Newton algorithm is closely related to the Kuo-Lu tree technique (see [15]). The Newton diagram of the substitution of the type $f(X, z + Y)$, $f \in \mathbb{C}\{X, Y\}$, $z \in \mathbb{C}\{X\}^*$ is analized in [13], [16]. The first author of this note applied the Newton algorithm in Cano's version to determine the so-called polar quotients with their multiplicities [19]. A survey of results concerning polar invariants (quotients) is given in [12]. The more general are the so-called jacobian quotients [17].

Now, let us introduce some definitions and facts similar to that from [19]. Let us consider the ring of Pusiseux polynomials $\mathbb{C}[X]^* = \bigcup_{N \geq 0} \mathbb{C}[X^{1/N}]$. For any $\varphi \in \mathbb{C}[X]^*$ we have $\deg \varphi < +\infty$. We put $\deg 0 = 0$. Since we consider only polynomials of positive orders this convention does not lead to a contradiction. Let $f \in \mathbb{C}\{X, Y\}$ be a reduced series such that the number $p = \operatorname{ord} f(0, Y) = (f, X)_0$ is finite and positive. We denote $f_\varphi := f(X, \varphi + Y) \in \mathbb{C}\{X^*, Y\}$. For any polynomial φ of positive order the diagram $\Delta f(X, \varphi + Y)$ has the vertex $(0, p)$ lying on the horizontal axis.

We denote by $\mathcal{N}(f, \varphi)$ the subset of the polygon $\mathcal{N}(f_\varphi)$ which consists segments with inclinations strictly greater than $\deg \varphi$. We define the hight of the polygon $|\mathcal{N}(f, \varphi)|$ as the sum of lengths of the projections of its segments on the vertical axis. The number of solutions $y \in \operatorname{Zer} f$ of the form $y = \varphi + \dots$ (equivalently $\operatorname{ord}(y - \varphi) > \deg \varphi$) equals $|\mathcal{N}(f, \varphi)| + \delta(f_\varphi)$. If f is reduced then $\delta(f_\varphi) \in \{0, 1\}$.

Definition 2.2. We define the set $T(f, X)$ of tracks of the Newton algorithm for f as the minimal subset (in the sense of inclusion) of $\mathbb{C}[X]^*$ such that the following conditions are satisfied:

(I) $0 \in T(f,X)$,
(II) for any $\varphi \in T(f,X)$, if there exists $S \in \mathcal{N}(f,\varphi)$ then for every nonzero root aX^θ of the initial form $\mathrm{in}(f_\varphi, S)$ we have $\varphi + aX^\theta \in T(f,X)$.

We have the following two equivalent characterizations of the set $T(f,X)$. Let

$$T'(f,X) = \{\varphi \in \mathbb{C}[X]^* : \exists y \in \mathrm{Zer}\, f \text{ such that } \mathrm{ord}(y-\varphi) > \deg \varphi\}$$

and let

$$T''(f,X) = \{\varphi \in \mathbb{C}[X]^* : |\mathcal{N}(f,\varphi)| + \delta(f_\varphi) > 0\}\ .$$

Proposition 2.3. ([19], Proposition 3.11) *$T(f,X) = T'(f,X) = T''(f,X)$.*

The following notions are useful in the proof of main result in the last chapter. Now, let us assume that $f \in \mathbb{C}\{X,Y\}$ is reduced and singular. Let $\varphi \in T(f,X)$. Let us introduce a symbol for the system of initial forms of solutions corresponding to $\mathcal{N}(f,\varphi)$ and $\delta(f_\varphi)$. For $\varphi = 0$ such system appears in Theorem 2.1 (ii). We put

$$\mathcal{I}(f,\varphi) = \bigoplus_{S \in \mathcal{N}(f,\varphi)} Zer\, \mathrm{in}(f_\varphi, S)^\circ + \langle 0 : \delta(f_\varphi)\rangle\ . \tag{7}$$

Clearly $\deg \mathcal{I}(f,\varphi) = |\mathcal{N}(f,\varphi)| + \delta(f_\varphi)$.

Definition 2.4. We say that a solution $y \in \mathrm{Zer}\, f$ is counted by a track $\varphi \in T(f,X)$ if all the conditions are satisfied:

(1) $\deg \mathcal{I}(f,\varphi) \geq 2$,
(2) $\mathrm{ord}(y-\varphi) > \deg \varphi$,
(3) $\mathrm{in}(y-\varphi) \in \mathcal{I}(f,\varphi)$,
(4) $\mathrm{in}(y-\varphi)$ has the multiplicity one in $\mathcal{I}(f,\varphi)$.

The following property is important.

Property 2.5. *Every $y \in \mathrm{Zer}\, f$ is counted by the unique $\varphi \in T(f)$.*

We denote this unique track by $\varphi = \varphi_f(y)$.

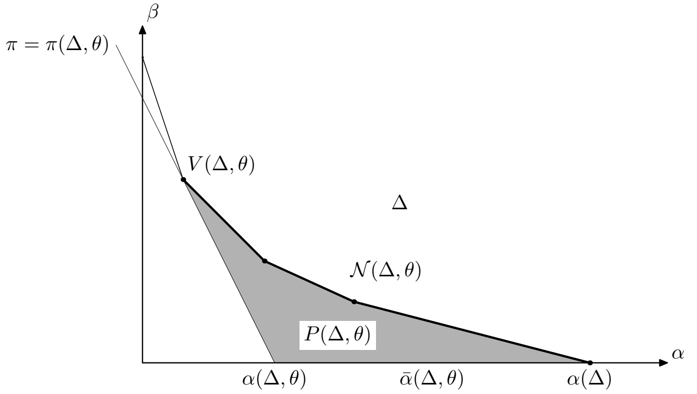
Example 2.6. Let $f = Y(Y-X)(Y-X-X^2)$. We have $\mathrm{Zer}\, f = \langle 0, X, X+X^2\rangle$ and $\varphi_f(0) = 0$, $\varphi_f(X) = X$, $\varphi_f(X+X^2) = X$.

3. Version of Kouchnirenko theorem

In this chapter we compute the Milnor number by using the Newton algorithm in Cano's version. Our main reference is [19]. Analogous results were obtained by Pi. Cassou-Noguès and Płoski in [6] (they applied the classical Newton algorithm) and by Gwoździewicz [11] who used the toric modification technique.

Let Δ be the Newton diagram of a nonzero series of $\mathbb{C}\{X^*,Y\}$ and let $\mathcal{N} = \mathcal{N}(\Delta)$ be the Newton polygon of this diagram. Let us denote by $\delta(\Delta)$ the distance between Δ and the horizontal axis. We consider only diagrams touching the vertical axis and with $\delta(\Delta) \leq 1$. With the above assumptions we have $\delta(\Delta) \in \{0,1\}$. For $\theta \geq 0$

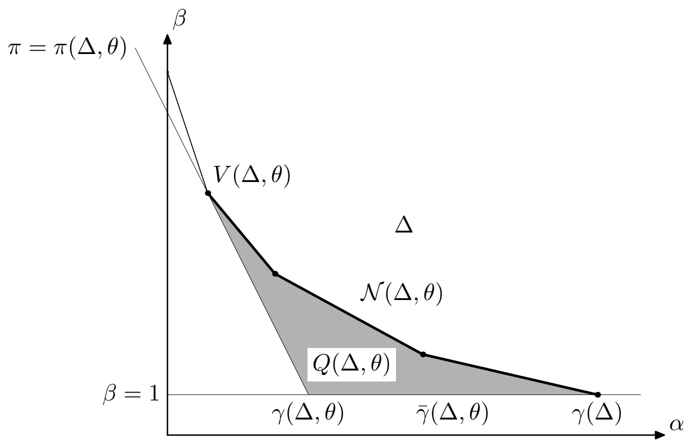
we define the stright line π with inclinaction θ that supports the diagram Δ. We denote this line by $\pi = \pi(\Delta, \theta)$. Let V be the commont point of π with Δ of the minimal possible ordinate. We denote this point by $V = V(\Delta, \theta)$; V must be a vertex of the diagram Δ.



The line $\pi(\Delta, \theta)$ crosses the horizontal axis at the point with abscissa $\alpha(\Delta, \theta) \geq 0$. Let $\mathcal{N} = \mathcal{N}(\Delta, \theta)$ denotes the subset of these segments of the Newton polygon $\mathcal{N}$ that have the inclinations strictly greater than θ. If the diagram Δ touches the horizontal axis ($\delta(\Delta) = 0$) then we define $\alpha(\Delta)$ as the minimal possible abscissa of the points of the diagram Δ that lie on the horizontal axis. Clearly $\alpha(\Delta, \theta) \leq \alpha(\Delta)$. We put

$$\bar{\alpha}(\Delta, \theta) = \alpha(\Delta) - \alpha(\Delta, \theta) .$$

If $\bar{\alpha}(\Delta, \theta) > 0$ then we define $P(\Delta, \theta)$ as the area of the polygon bounded by the line $\pi(\Delta, \theta)$, the polygon $\mathcal{N}(\Delta, \theta)$ and the horizontal axis. Otherwise, we put $P(\Delta, \theta) = 0$.



If the diagram Δ does not touch the horizontal axis ($\delta(\Delta) = 1$) then the line $\pi(\Delta, \theta)$ crosses the line $\beta = 1$ at the point with abscissa $\gamma(\Delta, \theta) \geq 0$. We define $\gamma(\Delta)$ to be the minimal abscissa of the points of the diagram Δ lying on the line $\beta = 1$. Clearly $\gamma(\Delta,\theta) \le \gamma(\Delta)$. We put

$$\bar{\gamma}(\Delta,\theta) = \gamma(\Delta) - \gamma(\Delta,\theta)\ .$$

If $\bar{\gamma}(\Delta,\theta) > 0$ then we define $Q(\Delta,\theta)$ as the area of the polygon bounded by the line $\pi(\Delta,\theta)$, the polygon $\mathcal{N}(\Delta,\theta)$ and by the line $\beta = 1$. Otherwise, we put $Q(\Delta,\theta) = 0$. If $\delta(\Delta) = 0$ then the numbers $\gamma(\Delta)$, $\gamma(\Delta,\theta)$, $\bar{\gamma}(\Delta,\theta)$ and $Q(\Delta,\theta)$ can be also defined assuming that the ordinate of the vertex $V(\Delta,\theta)$ is greater or equal to 1. Using the formula for area of triangle, we check that

$$2P(\Delta,\theta) - \bar{\alpha}(\Delta,\theta) = 2Q(\Delta,\theta) + \bar{\gamma}(\Delta,\theta)\ .$$

Now, let us discuss the notions introduced above in the context of the Newton algorithm. We assume that the series $f \in \mathbb{C}\{X,Y\}$ is reduced and that the number $p = (f,X)_0$ is finite and greater then one. We put

$$\hat{\mu}(f,\varphi) = \begin{cases} 2\,P(\Delta f_\varphi, \deg\varphi) - \bar{\alpha}(\Delta f_\varphi, \deg\varphi) & \text{if } \delta(f_\varphi) = 0\ , \\ 2\,Q(\Delta f_\varphi, \deg\varphi) + \bar{\gamma}(\Delta f_\varphi, \deg\varphi) & \text{if } \delta(f_\varphi) = 1\ . \end{cases}$$

Theorem 3.1. *With the above assumptions on f*

(a) *for almost all $\varphi \in T(f,X)$ the number $\hat{\mu}(f,\varphi)$ equals zero,*
(b) $\mu(f) = 1 - p + \sum_{\varphi\in T(f,X)} \hat{\mu}(f,\varphi)$.

Proof. Let us recall few notions [19]. For a series $g \in \mathbb{C}\{X^*,Y\}$ and for a segment S of its Newton polygon we denote by $t(g,S)$ the number of different roots of initial form $\mathrm{in}(g,S)$ in $\mathbb{C}\{X\}^*$. Let $r_1,\dots,r_s$ denote the multiplicities of nonzero roots among all these roots ($t-1 \le s \le t$). Clearly $r_1+\cdots+r_s = |S|_{\mathbf{V}}$. We put $d(g,S) = (r_1-1)+\cdots+(r_s-1)$ and we call $d(g,S)$ the degeneracy of g on S. The condition $d(g,S) = 0$ means that every nonzero root is a single root (nondegeneracy). We have

$$t(g,S) - 1 + d(g,S) = |S|_{\mathbf{V}} + \varepsilon(S) \tag{8}$$

where $\varepsilon(S) = -1$ for a segment S touching the horizontal axis and $\varepsilon(S) = 0$ for segments that do not touch the horizontal axis. The number $\alpha(S)$ equals the abscissa of point where the line containing segment S crosses the horizontal axis.

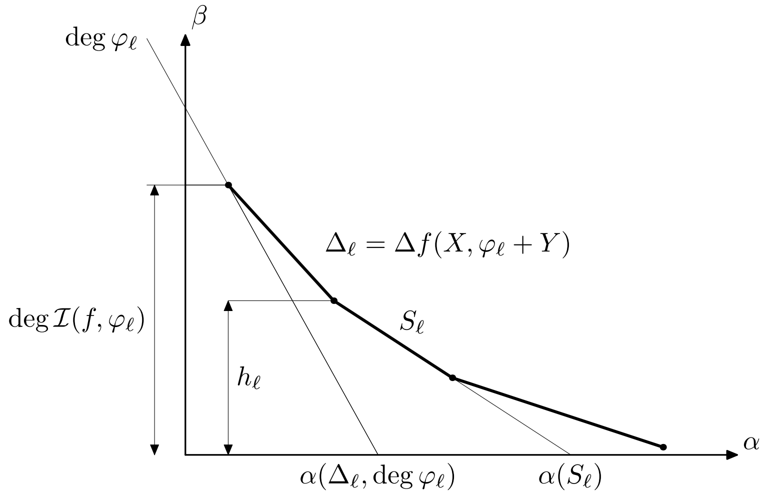
We apply the following fact.

Proposition 3.2. ([19], Proposition 3.9) *Let us assume that $\varphi \in \mathbb{C}[X]^*$ is a polynomial such that the polygon $\mathcal{N}(f,\varphi)$ is nonempty. Let $S \in \mathcal{N}(f,\varphi)$ and let aX^θ be a nonzero root of the form* $\mathrm{in}(f_\varphi,S)$. *Then*

$$\deg\mathcal{I}(f,\varphi+aX^\theta) = \textit{multiplicity of } aX^\theta \textit{ as a root of the form } \mathrm{in}(f_\varphi,S)\ .$$

Proof of (a). We base on [19]. Let $y = a_1X^{\theta_1} + a_2X^{\theta_2} + \dots$ ($a_1,a_2,\dots$ nonzero, $0 < \theta_1 < \theta_2 < \dots$) be a Puiseux solution of the equation $f = 0$ in $\mathbb{C}\{X\}^*$. Without loss of generality it suffices to consider a solution with infinite number of terms. We define tracks $\varphi_1 = 0$ and $\varphi_\ell = a_1X^{\theta_1}+\cdots+a_{\ell-1}X^{\theta_{\ell-1}}$ for $\ell = 2,3,\dots$. Let

$\Delta_\ell := \Delta f(X, \varphi_\ell + Y)$. Let us fix $\ell \in \{1, 2, \dots\}$. According to the Newton algorithm there exists a segment S_ℓ of the polygon $\mathcal{N}(f, \varphi_\ell)$ such that $a_\ell X^{\theta_\ell}$ is a root of the form $\mathrm{in}(f_{\varphi_\ell}, S_\ell)$. We denote $h_\ell = \deg_Y \mathrm{in}(f_{\varphi_\ell}, S_\ell)$.



Let r_ℓ be the multiplicity of the root. Obviously $h_\ell \geq r_\ell$. By Proposition 3.2 $r_\ell = \deg \mathcal{I}(f, \varphi_\ell + a_\ell X^{\theta_\ell}) = |\mathcal{N}(f, \varphi_{\ell+1})| + \delta(f_{\varphi_{\ell+1}}) \geq \deg_Y \mathrm{in}(f_{\varphi_{\ell+1}}, S_{\ell+1}) = h_{\ell+1}$. This construction gives the infinite sequence of positive integers $h_1 \geq r_1 \geq h_2 \geq r_2 \geq \dots$ that must stabilize. The stable value is the multiplicity of y as a root of f. For the reduced series it equals one. Let us note that the equality $h_\ell = r_\ell$ means that the segment S_ℓ touches the horizontal axis and that $a_\ell X^{\theta_\ell}$ is the unique root of the initial form $\mathrm{in}(f_{\varphi_\ell}, S_\ell)$. Then $t(f_{\varphi_\ell}, S_\ell) = 1$ which will be important in the proof of part (b). Moreover, from the step where stability is reached, we will have $|\mathcal{N}(f, \varphi_\ell)| = 1$. Then we get $\hat{\mu}(f, \varphi_\ell) = 0$ for such terms.

Proof of (b). Applying the Teissier Lemma (cited and proved e.g. in [6]) we have

$$\mu(f) = 1 - p + \left(f, \frac{\partial f}{\partial Y}\right)_0 = 1 - p + \sum_{j=1}^{p-1} \mathrm{ord}\, f(X, z_j(X)) \tag{9}$$

where $z_1, \dots, z_{p-1} \in \mathbb{C}\{X\}^*$ is the sequence of solutions of the equation $(\partial f/\partial Y) = 0$. The system

$$\langle \mathrm{ord}\, f(X, z_1(X)), \dots, \mathrm{ord}\, f(X, z_{p-1}(X)) \rangle$$

giving the so-called polar quotients was described in [19] (Theorem 2.1). Using this result we can write the equality (9) as

$$\mu(f) = 1 - p + \sum_{\varphi \in T(f,X)} \sum_{S \in \mathcal{N}(f,\varphi)} \alpha(S)[t(f_\varphi, S) - 1]\,. \tag{10}$$

In the proof of part (a) we checked that almost all components in the above sum equal zero. Now, to finish the proof it suffices to show that (10) equals the right side in the statement (b) of the theorem. In order to simplify notation we hide the dependence of f as in the following table.

new symbol	instead of
$\mathcal{N}_\varphi$	$\mathcal{N}(f,\varphi)$
$\hat\mu_\varphi$	$\hat\mu(f,\varphi)$
δ_φ	$\delta(f_\varphi)$
π_φ	$\pi(\Delta f_\varphi, \deg\varphi)$
V_φ	$V(\Delta f_\varphi, \deg\varphi)$
P_φ	$P(\Delta f_\varphi, \deg\varphi)$
Q_φ	$Q(\Delta f_\varphi, \deg\varphi)$

Let n_φ be the number of segments of polygon $\mathcal{N}_\varphi$ ($n_\varphi \geq 0$). We number the segments of $\mathcal{N}_\varphi$ from up to down:

$$S_\varphi^{(1)}, \ldots, S_\varphi^{(n_\varphi)} .$$

For $i = 1, \ldots, n_\varphi$ we put $t_\varphi^{(i)} := t(f_\varphi, S^{(i)})$, $d_\varphi^{(i)} := d(f_\varphi, S^{(i)})$, $\alpha_\varphi^{(i)} := \alpha(S^{(i)})$, $\varepsilon_\varphi^{(i)} := \varepsilon(S^{(i)})$. Moreover $\alpha_\varphi^{(0)} := \alpha(\Delta f_\varphi, \deg\varphi)$ and $\bar\alpha_\varphi = \alpha_\varphi^{(n_\varphi)} - \alpha_\varphi^{(0)}$.

Applying (8) and denoting $b_\varphi^{(i)} = |S_\varphi^{(i)}|_{\mathbf{V}}$ ($i = 1, \ldots, n_\varphi$) we can write

$$t_\varphi^{(i)} - 1 + d_\varphi^{(i)} = b_\varphi^{(i)} + \varepsilon_\varphi^{(i)} \text{ for } i = 1, \ldots, n_\varphi . \tag{11}$$

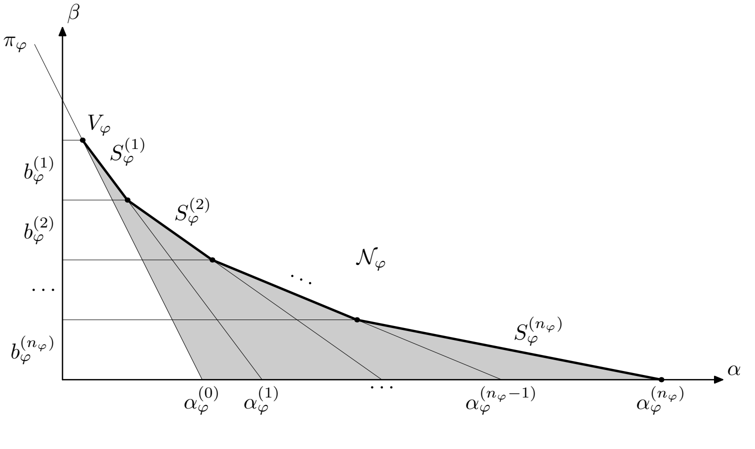
The formula (10) can be rewritten as

$$\mu(f) = 1 - p + \sum_{\varphi\in T(f,X)} \sum_{i=1}^{n_\varphi} \alpha_\varphi^{(i)} (t_\varphi^{(i)} - 1) . \tag{12}$$

Let us fix $\varphi \in T(f,X)$. We are going to prove that

$$\sum_{i=1}^{n_\varphi} \alpha_\varphi^{(i)} (t_\varphi^{(i)} - 1) = \hat\mu_\varphi + \alpha_\varphi^{(0)} (|\mathcal{N}_\varphi| + \delta_\varphi - 1) - \sum_{i=1}^{n_\varphi} \alpha_\varphi^{(i)} d_\varphi^{(i)} . \tag{13}$$

First, we consider the case $\delta_\varphi = 0$.



We have

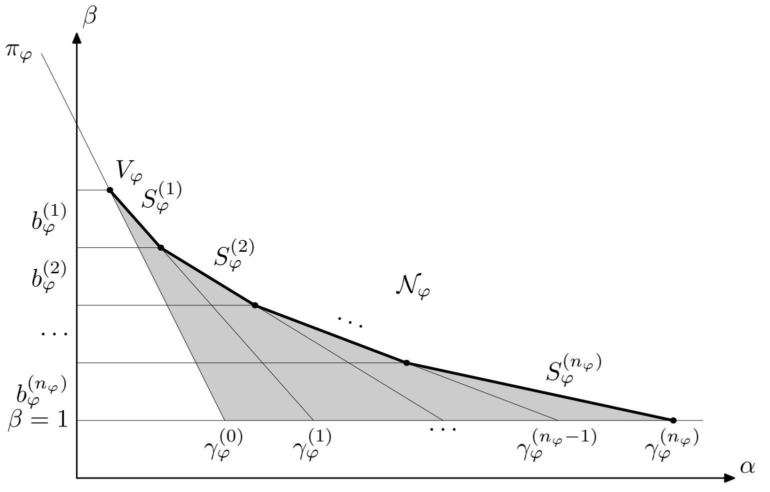
$$(14) \qquad 2\,P_\varphi = \sum_{i=1}^{n_\varphi} (\alpha_\varphi^{(i)} - \alpha_\varphi^{(i-1)})(b_\varphi^{(i)} + \cdots + b_\varphi^{(n_\varphi)}) = -\alpha_\varphi^{(0)}|\mathcal{N}_\varphi| + \sum_{i=1}^{n_\varphi} \alpha_\varphi^{(i)} b_\varphi^{(i)}\ .$$

By (11) and (14) we can write

$$\begin{aligned}\sum_{i=1}^{n_\varphi} \alpha_\varphi^{(i)}(t_\varphi^{(i)} - 1) &= \sum_{i=1}^{n_\varphi} \alpha_\varphi^{(i)}(b_\varphi^{(i)} + \varepsilon_\varphi^{(i)} - d_\varphi^{(i)}) = \sum_{i=1}^{n_\varphi} \alpha_\varphi^{(i)}(b_\varphi^{(i)} - d_\varphi^{(i)}) - \alpha_\varphi^{(n_\varphi)} \\ &= \sum_{i=1}^{n_\varphi} \alpha_\varphi^{(i)} b_\varphi^{(i)} - \sum_{i=1}^{n_\varphi} \alpha_\varphi^{(i)} d_\varphi^{(i)} - \alpha_\varphi^{(n_\varphi)} \\ &= \hat{\mu}_\varphi + \alpha_\varphi^{(0)}(|\mathcal{N}_\varphi| - 1) - \sum_{i=1}^{n_\varphi} \alpha_\varphi^{(i)} d_\varphi^{(i)}\end{aligned}$$

which gives (13).

Now, let us check the case $\delta_\varphi = 1$. For $i = 1, \ldots, n_\varphi$ we put $\gamma_\varphi^{(i)} := \gamma(S^{(i)})$. Moreover $\gamma_\varphi^{(0)} = \gamma(\Delta f_\varphi, \deg\varphi)$ and $\bar{\gamma}_\varphi = \gamma_\varphi^{(n_\varphi)} - \gamma_\varphi^{(0)}$.



Let us note that for $i = 1, \ldots, n_\varphi$ (by the formula for area of triangle)

$$(\gamma_\varphi^{(i)} - \gamma_\varphi^{(i-1)})(b_\varphi^{(i)} + \cdots + b_\varphi^{(n_\varphi)}) + (\gamma_\varphi^{(i)} - \gamma_\varphi^{(i-1)}) = (\alpha_\varphi^{(i)} - \alpha_\varphi^{(i-1)})(b_\varphi^{(i)} + \cdots + b_\varphi^{(n_\varphi)})\ .$$

By using the above observation we get

$$\begin{aligned}\hat{\mu}_\varphi = 2Q_\varphi + \bar{\gamma}_\varphi &= \sum_{i=1}^{n_\varphi} (\gamma_\varphi^{(i)} - \gamma_\varphi^{(i-1)})(b_\varphi^{(i)} + \cdots + b_\varphi^{(n_\varphi)}) + \sum_{i=1}^{n_\varphi} (\gamma_\varphi^{(i)} - \gamma_\varphi^{(i-1)}) \\ &= \sum_{i=1}^{n_\varphi} (\alpha_\varphi^{(i)} - \alpha_\varphi^{(i-1)})(b_\varphi^{(i)} + \cdots + b_\varphi^{(n_\varphi)}) = -\alpha_\varphi^{(0)}|\mathcal{N}_\varphi| + \sum_{i=1}^{n_\varphi} \alpha_\varphi^{(i)} b_\varphi^{(i)}\ .\end{aligned}$$

Now, we compute

$$\sum_{i=1}^{n_\varphi}\alpha_\varphi^{(i)}(t_\varphi^{(i)}-1)=\sum_{i=1}^{n_\varphi}\alpha_\varphi^{(i)}(b_\varphi^{(i)}-d_\varphi^{(i)})=\sum_{i=1}^{n_\varphi}\alpha_\varphi^{(i)}b_\varphi^{(i)}-\sum_{i=1}^{n_\varphi}\alpha_\varphi^{(i)}d_\varphi^{(i)}$$
$$=\hat{\mu}_\varphi+\alpha_\varphi^{(0)}|\mathcal{N}_\varphi|-\sum_{i=1}^{n_\varphi}\alpha_\varphi^{(i)}d_\varphi^{(i)}\ ,$$

which also gives (13).

Applying (12) and by using (13) we get

$$\mu(f)=1-p+\sum_{\varphi\in T(f,X)}\sum_{i=1}^{n_\varphi}\alpha_\varphi^{(i)}(t_\varphi^{(i)}-1)$$
$$=1-p+\sum_{\varphi\in T(f,X)}\hat{\mu}_\varphi+\sum_{\varphi\in T(f,X)}\alpha_\varphi^{(0)}(|\mathcal{N}_\varphi|+\delta_\varphi-1)-\sum_{\varphi\in T(f,X)}\sum_{i=1}^{n_\varphi}\alpha_\varphi^{(i)}d_\varphi^{(i)}\ .$$

Therefore, to the finish of the proof it suffices to show that

$$\sum_{\varphi\in T(f,X)}\sum_{i=1}^{n_\varphi}\alpha_\varphi^{(i)}d_\varphi^{(i)}=\sum_{\varphi\in T(f,X)}\alpha_\varphi^{(0)}(|\mathcal{N}_\varphi|+\delta_\varphi-1)\ . \tag{15}$$

We denote by $T_\ell(f,X)$ the set of all tracks with the lenght ℓ ($\ell=0,1,2,\dots$). These sets are finite. The set $T_0(f,X)$ contains only zero track. For $\varphi=0$ we have $\alpha_\varphi^{(0)}=0$. Hence, the component on the right side of the formula (15) corresponding to zero track equals zero. Therefore, it is enough to show

$$\sum_{\varphi\in T_\ell(f,X)}\sum_{i=1}^{n_\varphi}\alpha_\varphi^{(i)}d_\varphi^{(i)}=\sum_{\varphi\in T_{\ell+1}(f,X)}\alpha_\varphi^{(0)}(|\mathcal{N}_\varphi|+\delta_\varphi-1) \tag{16}$$

for $\ell=0,1,2,\dots$. Let us fix $\varphi\in T_\ell(f,X)$. To this track we can assign the tracks of the form $\varphi+aX^\theta\in T_{\ell+1}(f,X)$ taking as aX^θ all different nonzero roots of all forms

$$\mathrm{in}(f_\varphi,S_\varphi^{(1)}),\dots,\mathrm{in}(f_\varphi,S_\varphi^{(n_\varphi)})\ .$$

We write these roots as $a_{ij}X^{\theta_i}$ ($j=1,\dots,s_\varphi^{(i)}$, $i=1,\dots,n_\varphi$), remembering about the dependence of coefficients and exponents on φ; $s_\varphi^{(i)}:=t_\varphi^{(i)}-1-\varepsilon_\varphi^{(i)}$ stands for the number of different nonzero roots of the form $\mathrm{in}(f_\varphi,S_\varphi^{(i)})$. For $\varphi\in T_\ell(f,X)$ ($\ell\geq 0$) we can write

$$T_{\ell+1}(\varphi)=\{\varphi+a_{ij}X^{\theta_i}\,:\,i=1,\dots,n_\varphi\,j=1,\dots,s_\varphi^{(i)}\}\ .$$

If $T_\ell(f,X)=\{\varphi_1,\dots,\varphi_m\}$ ($m\geq 1$), then $T_{\ell+1}(f,X)=T_{\ell+1}(\varphi_1)\cup\dots\cup T_{\ell+1}(\varphi_m)$. Hence, it suffices to check (16) taking into consideration fixed track $\varphi\in T_\ell(f,X)$ on the left side, while on the right side the set $T_{\ell+1}(\varphi)$. The appropriate formula has the form

$$(17)\qquad \sum_{i=1}^{n_\varphi} \alpha_\varphi^{(i)} d_\varphi^{(i)} = \sum_{i=1}^{n_\varphi} \sum_{j=1}^{s_\varphi^{(i)}} \alpha^{(0)}_{\varphi+a_{ij}X^{\theta_i}} (|\mathcal{N}_{\varphi+a_{ij}X^{\theta_i}}| + \delta_{\varphi+a_{ij}X^{\theta_i}} - 1)\ .$$

The property of the Newton algorithm implies that $\alpha_\varphi^{(i)} = \alpha^{(0)}_{\varphi+a_{ij}X^{\theta_i}}$. Therefore for the proof of the above equality it is enough to show that

$$(18)\qquad d_\varphi^{(i)} = \sum_{j=1}^{t_\varphi^{(i)}} (|\mathcal{N}_{\varphi+a_{ij}X^{\theta_i}}| + \delta_{\varphi+a_{ij}X^{\theta_i}} - 1)\ .$$

Let $r_\varphi^{(i,j)}$ be the multiplicity of $a_{ij}X^{\theta_j}$ as a root of the form $\mathrm{in}(f_\varphi, S_\varphi^{(i)})$. Then

$$d_\varphi^{(i)} = \sum_{j=1}^{s_\varphi^{(i)}} (r_\varphi^{(i,j)} - 1)\ .$$

Therefore, for the proof of (18) it suffices to know that

$$r_\varphi^{(i,j)} = |\mathcal{N}_{\varphi+a_{ij}X^{\theta_i}}| + \delta_{\varphi+a_{ij}X^{\theta_i}}\ ,$$

but it follows directly from Proposition 3.2. □

4. Proof of Gusein-Zade Theorem

Let $f \in \mathbb{C}\{X,Y\}$ be a reduced and singular series. In analogy to the set $T(f,X)$ of tracks of the Newton algorithm discussed in Section 2 we define below a new set $T_*(f,X) \subset T(f,X)$ which is finite an can be applied to compute the Milnor number by Theorem 3.1.

Definition 4.1. We define the set $T_*(f,X)$ of multiple tracks of the Newton algorithm for f as the minimal subset (in the sense of inclusion) of $\mathbb{C}[X]^*$ such that the following conditions are satisfied:

(I) $0 \in T_*(f,X)$,
(II) for any $\varphi \in T_*(f,X)$, if there exists $S \in \mathcal{N}(f,\varphi)$ then for every nonzero multiple root aX^θ of the initial form $\mathrm{in}(f_\varphi, S)$ we have $\varphi + aX^\theta \in T_*(f,X)$.

In analogy to $T(f,X)$ the set $T_*(f,X)$ has also two equivalent characterizations. Let

$$T'_*(f,X) = \{\varphi \in \mathbb{C}[X]^* : \exists y^{(1)} \neq y^{(2)} \in \mathrm{Zer}\, f \text{ that } \mathrm{ord}(y^{(i)} - \varphi) > \deg\varphi,\ i = 1,2\}$$

and let

$$T''_*(f,X) = \{\varphi \in \mathbb{C}[X]^* : |\mathcal{N}(f,\varphi)| + \delta(f_\varphi) > 1\}\ .$$

Proposition 4.2. $T_*(f,X) = T'_*(f,X) = T''_*(f,X)$.

The proof is analogous to the proof of Proposition 2.3.

Proposition 4.3. *Let $\varphi \in T(f,X)$. Then $\varphi \in T_*(f,X)$ if and only if $\hat{\mu}(f,\varphi) > 0$.*

Below we present the steps of construction a deformation

$$F_t = f + tX^{\omega_0} f_1^{\omega_1} \dots f_{\mathbf{h}}^{\omega_{\mathbf{h}}} \tag{19}$$

where f has the form (3) and f_k $(k = 1,\dots,\mathbf{h})$ are defined in (5). All what can be controlled are nonegative integers $\omega_0, \omega_1, \dots, \omega_{\mathbf{h}}$. Applying Treorem 3.1 and Proposition 4.3 we can write

$$\mu(f) = 1 - (f,X)_0 + \sum_{\varphi \in T_*(f,X)} \hat{\mu}(f,\varphi)\ , \tag{20}$$

$$\mu(F_t) = 1 - (F_t,X)_0 + \sum_{\varphi \in T_*(F_t,X)} \hat{\mu}(F_t,\varphi)\ . \tag{21}$$

Since we want $\mu(f)$ and $\mu(F_f)$ to be close, the idea in choosing $\omega_0, \omega_1, \dots, \omega_{\mathbf{h}}$ is to obtain many common elements in both (20) and (21). To have equality $(f,X)_0 = (F_f,X)_0$ it suffices that $\omega_0 > 0$. Moreover, we want to have as many common tracks as possible. For example, the equality holds $T_*(f,X) = T_*(F_t,X) = \{0\}$ in Bodin's deformation from Introduction. In our construction we will obtain $T_*(F_t,X) \subset T_*(f,X)$. Unfortunately, the inclusion may be strict.

As in Introduction we apply that f is generated by a cycle of $y \in \mathbb{C}\{X\}^*$ (1) in the sense of (3). On the basis of y we can define tracks: $\varphi_1 := 0$, $\varphi_\ell := a_1 X^{v_1/v_0} + \dots + a_{\ell-1} X^{v_{\ell-1}/v_0}$ $(\ell = 2,3,\dots)$. By Proposition 2.3 we have $T(f,X) = \mathrm{cycle}(\varphi_1) \cup \mathrm{cycle}(\varphi_2) \cup \dots$. In order to determine $T_*(f,X)$ let us recall a description of the Newton polygon $\mathcal{N}(f,\varphi_\ell)$ from [19]. The notation is equivalent. We put $w^* = \mathrm{GCD}(v_0, v_1, \dots)$.

Property 4.4. ([19], Property 5.1)

- (i) *Polygon $\mathcal{N}(f,\varphi_\ell)$ consists one segment S_ℓ with inclination v_ℓ/v_0 which touches the horizontal axis,*
- (ii) $\deg_Y \mathrm{in}(f_{\varphi_\ell}, S_\ell) = \mathrm{GCD}(v_0,\dots,v_{\ell-1})/w^*$.
- (iii) *Every root of* $\mathrm{in}(f_{\varphi_\ell}, S_\ell)$ *has the multiplicity* $\mathrm{GCD}(v_0,\dots,v_\ell)/w^*$,
- (iv) $t(f_{\varphi_\ell}, S_\ell) = \frac{\mathrm{GCD}(v_0,\dots,v_{\ell-1})}{\mathrm{GCD}(v_0,\dots,v_\ell)}$.

In addition to Property 4.4 we will need more precise information about the initial form $\mathrm{in}(f_{\varphi_\ell}, S_\ell)$. Let $w_\ell = \mathrm{GCD}(v_0,\dots,v_\ell)$, $u_\ell = w_{\ell-1}/w_\ell$, $\theta_\ell = v_\ell/v_0$.

Property 4.5. *With the previous notation there exist $c \neq 0$ and $\zeta \geq 0$ such that*

$$\mathrm{in}(f_{\varphi_\ell}, S_\ell) = cX^\zeta (Y^{u_\ell} - a_\ell^{u_\ell} X^{\theta_\ell u_\ell})^{w_\ell/w^*}\ .$$

Proof. See (e.g. [20], Lemma 6.1).

Let us return to tracks. Since $\mathrm{GCD}(v_0,\dots,v_{\ell_{\mathbf{h}}})/w^* = e_{\mathbf{h}} = 1$ then it follows from Property 4.4 (iii) that every root of the corresponding initial form is a single root. Therefore a track $\varphi_{\ell_{\mathbf{h}}} = y_{\mathbf{h}} \in T_*(f,X)$ cannot be extended in the sense of

Definition 4.1. Hence $T_*(f,X) = \mathrm{cycle}(\varphi_1) \cup \cdots \cup \mathrm{cycle}(\varphi_{\ell_{\mathbf{h}}})$. Our effort was to obtain the equality $T_*(F_t,X) = T_*(f,X)$. However, we finished with the following two cases:

(I) $T_*(F_t,X) = \mathrm{cycle}(\varphi_1) \cup \cdots \cup \mathrm{cycle}(\varphi_{\ell_{\mathbf{h}}})$,
(II) $T_*(F_t,X) = \mathrm{cycle}(\varphi_1) \cup \cdots \cup \mathrm{cycle}(\varphi_{\ell_{\mathbf{h}}-1})$.

In both cases we want for $\ell < \ell_{\mathbf{h}}$ to have $\hat{\mu}(F_t,\varphi_\ell) = \hat{\mu}(f,\varphi_\ell)$. When $\ell = \ell_{\mathbf{h}}$ we want $\hat{\mu}(F_t,\varphi_{\ell_{\mathbf{h}}}) = \hat{\mu}(f,\varphi_{\ell_{\mathbf{h}}}) - \frac{1}{N_{\mathbf{h}-1}}$ in the first case and $\hat{\mu}(f,\varphi_{\ell_{\mathbf{h}}}) = \frac{1}{N_{\mathbf{h}-1}}$ in the second case. Since $\#\mathrm{cycle}(\varphi_{\ell_{\mathbf{h}}}) = N_{\mathbf{h}-1}$ this will give $\mu(F_t) = \mu(f) - 1$ in both cases.

In order to describe the diagrams $\Delta f(X,\varphi_\ell + Y)$ and $\Delta F_t(X,\varphi_\ell + Y)$ we need the shapes of the diagrams $\Delta f(X,\varphi_\ell + Y)$ for $\ell = 1,\ldots,\ell_{\mathbf{h}}$ and $\Delta f_k(X,\varphi_\ell + Y)$ for $k = 1,\ldots,\mathbf{h}$ and $\ell = 1,\ldots,\ell_{\mathbf{h}}$. To this end let us recall facts from [13]. The contact exponent between the branch f and an arbitrary Puiseux series $z \in \mathbb{C}\{X\}^*$ is defined as

$$o_f(z) = \max\{\mathrm{ord}(z - \tau^0 * y), \ldots, \mathrm{ord}(z - \tau^{N-1} * y)\}\ . \tag{22}$$

Below, we describe the shapes of the diagrams by using the so-called Teissier notation. For $A, B \subset \mathbb{R}^2_+$ $A + B = \{a + b :\ a \in A,\ b \in B\}$, $\left\{\frac{a}{b}\right\}$ = the convex hull of $\{(a,0),(0,b)\} + \mathbb{R}^2_+$. Moreover $\{\frac{1}{\infty}\} = (1,0) + \mathbb{R}^2_+$ and $\{\frac{\infty}{1}\} = (0,1) + \mathbb{R}^2_+$ ($\left\{\frac{0}{0}\right\}$ is the identity). By convention the sum over the empty set equals $\{\frac{0}{0}\}$.

Property 4.6. (Properties 3.1 and 3.2 in [13])
Let $(b_0, b_1, \ldots, b_{\mathbf{h}})$ be the characteristic sequence of the branch.

(I) *If there exists the smallest integer k such that $o_f(z) \le b_k/b_0$ then*

$$\Delta f(X, z+Y) = \sum_{j=1}^{k-1} \left\{\frac{(b_j/b_0)(e_{j-1}-e_j)}{e_{j-1}-e_j}\right\} + \left\{\frac{o_f(z)\, e_{k-1}}{e_{k-1}}\right\}\ .$$

(II) *If $b_{\mathbf{h}}/b_0 < o_f(z)$ then*

$$\Delta f(X, z+Y) = \sum_{k=1}^{\mathbf{h}} \left\{\frac{(b_k/b_0)(e_{k-1}-e_k)}{e_{k-1}-e_k}\right\} + \left\{\frac{o_f(z)}{1}\right\}\ .$$

Corollary 4.7. (for f and φ_ℓ) We have $o_f(\varphi_\ell) = v_\ell/v_0$. Therefore:

(I) if there exists the smallest integer k such that $\ell \le \ell_k$ then

$$\Delta f(X, \varphi_\ell+Y) = \sum_{j=1}^{k-1} \left\{\frac{(b_j/b_0)(e_{j-1}-e_j)}{e_{j-1}-e_j}\right\} + \left\{\frac{(v_\ell/v_0)\, e_{k-1}}{e_{k-1}}\right\}\ ,$$

(II) if $\ell_{\mathbf{h}} < \ell$ then

$$\Delta f(X, \varphi_\ell+Y) = \sum_{k=1}^{\mathbf{h}} \left\{\frac{(b_k/b_0)(e_{k-1}-e_k)}{e_{k-1}-e_k}\right\} + \left\{\frac{v_\ell/v_0}{1}\right\}\ .$$

We can also describe $\Delta f_k(X,\varphi_\ell+Y)$ $(k=1,\dots,\mathbf{h})$. The characteristic sequence of f_k has the form: $(b_0/e_{k-1}, b_1/e_{k-1},\dots,b_{k-1}/e_{k-1})$ with the first sequence of divisors: $(e_0/e_{k-1}, e_1/e_{k-1},\dots,e_{k-1}/e_{k-1})$. Let us observe that

$$o_{f_k}(\varphi_\ell)=\begin{cases} v_\ell/v_0 & \text{for } \ell<\ell_k \\ +\infty & \text{for } \ell=\ell_k \\ b_k/b_0 & \text{for } \ell_k<\ell \end{cases}. \tag{23}$$

Corollary 4.8. (for f_k and φ_ℓ)

(I) If $\ell<\ell_k$ then there exists the smallest integer $j\in\{1,\dots,k\}$ such that $\ell\le\ell_j$. Then

$$\Delta f_k(X,\varphi_\ell+Y)=\sum_{i=1}^{j-1}\left\{\frac{(b_i/b_0)(e_{i-1}/e_{k-1}-e_i/e_{k-1})}{e_{i-1}/e_{k-1}-e_i/e_{k-1}}\right\}+\left\{\frac{(v_\ell/v_0)\,e_{j-1}/e_{k-1}}{e_{j-1}/e_{k-1}}\right\}.$$

(II) If $\ell=\ell_k$ then

$$\Delta f_k(X,\varphi_{\ell_k}+Y)=\sum_{j=1}^{k-1}\left\{\frac{(b_j/b_0)(e_{j-1}/e_{k-1}-e_j/e_{k-1})}{e_{j-1}/e_{k-1}-e_j/e_{k-1}}\right\}+\left\{\frac{\infty}{1}\right\}.$$

(III) If $\ell_k<\ell$ then

$$\Delta f_k(X,\varphi_\ell+Y)=\sum_{j=1}^{k-1}\left\{\frac{(b_j/b_0)(e_{j-1}/e_{k-1}-e_j/e_{k-1})}{e_{j-1}/e_{k-1}-e_j/e_{k-1}}\right\}+\left\{\frac{b_k/b_0}{1}\right\}.$$

Below we apply the semigroup technique from [8]. Now, our aim is to construct $\omega_0,\dots,\omega_{\mathbf{h}}$ $(\mathbf{h}\ge1)$ by using the longest track $y_{\mathbf{h}}=\varphi_{\ell_{\mathbf{h}}}$ in $T_*(f,X)$. We will apply the semigroup generators $\bar b_0,\bar b_1,\dots,\bar b_{\mathbf{h}}$ which satisfy relations $\bar b_0=b_0$, $\bar b_1=b_1$, $\bar b_{k+1}=n_k\bar b_k+b_{k+1}-b_k$ for $k=1,\dots,\mathbf{h}-1$. It follows from the above relation that $n_k\bar b_k<\bar b_{k+1}$ for $k=1,\dots,\mathbf{h}-1$ (God given inequality).

The following proposition follows from Corollaries 4.7 and 4.8.

Proposition 4.9.

(i) *For $k=1,\dots,\mathbf{h}-1$ the diagram $\Delta f_k(X,y_{\mathbf{h}}+Y)$ has the vertex on the horizontal axis with abscissa $\bar b_k/b_0$.*

(ii) *The diagram $\Delta f_{\mathbf{h}}(X,y_{\mathbf{h}}+Y)$ does not touch the horizontal axis and its lower vertex (with ordinate one) has the abscissa $(n_{\mathbf{h}-1}\bar b_{\mathbf{h}-1}-b_{\mathbf{h}-1})/b_0$.*

(iii) *The last segment $S_{\mathbf{h}}$ of the diagram $\Delta f(X,y_{\mathbf{h}}+Y)$ has the inclination $|S_{\mathbf{h}}|_{\mathbf{H}}/|S_{\mathbf{h}}|_{\mathbf{V}}=b_{\mathbf{h}}/b_0$ and touches the horizontal axis at the point with abscissa $n_{\mathbf{h}}\bar b_{\mathbf{h}}/b_0$. The length of vertical projection is $|S_{\mathbf{h}}|_{\mathbf{V}}=n_{\mathbf{h}}$.*

(iv) *The straight line $\pi_{\mathbf{h}-1}$ determined by the penultimate segment of the diagram $\Delta f(X,\varphi_{\mathbf{h}}+Y)$ (the line and the segment have the inclination $b_{\mathbf{h}-1}/b_0$) crosses the horizontal axis at the point with abscissa $n_{\mathbf{h}}n_{\mathbf{h}-1}\bar b_{\mathbf{h}-1}/b_0$.*

Let us notice that all the series $f_1(X,\varphi_{\mathbf{h}}+Y)$, …, $f_{\mathbf{h}}(X,\varphi_{\mathbf{h}}+Y)$, $f(X,\varphi_{\mathbf{h}}+Y)$ are in the ring $\mathbb{C}\{X^{1/N_{\mathbf{h}-1}},Y\}$ where $N_{\mathbf{h}-1}=n_1\dots n_{\mathbf{h}-1}$. Hence, all the points corresponding to nonzero coefficients have the form

$$\left(\frac{i}{n_1 \ldots n_{\mathbf{h}-1}}, \beta\right) \tag{24}$$

for nonegative integers i, β. Now, let us consider nonegative integer numbers $\omega_0, \omega_1, \ldots, \omega_{\mathbf{h}}$. Let us denote

$$H = X^{\omega_0} f_1^{\omega_1} \ldots f_{\mathbf{h}}^{\omega_{\mathbf{h}}} \, . \tag{25}$$

The polynomial H depends on $\omega_0, \omega_1, \ldots, \omega_{\mathbf{h}}$ what is not explicitly written.

Lemma 4.10. *Let $B(\alpha, \beta)$ be a point of the form* (24) *lying over the straight line $\pi_{\mathbf{h}-1}$ or on this line in the belt $0 \le \beta < n_{\mathbf{h}}$. Then the numbers $\omega_0, \omega_1, \ldots, \omega_{\mathbf{h}}$ may be chosen with condition $0 \le \omega_k < n_k$ $(k = 1, \ldots, \mathbf{h})$ and such that the lowest vertex of the diagram $\Delta H(X, \varphi_{\mathbf{h}} + Y)$ equals B.*

Proof. From the fact that the diagram of the product equals the sum of diagrams of factors follows that the lowest vertex of the diagram $\Delta H(X, \varphi_{\mathbf{h}} + Y)$ is a linear combination of the lowest vertices of the diagrams $\Delta X = \{\frac{\infty}{1}\}$, $\Delta f_1(X, \varphi_{\mathbf{h}} + Y)$, $\ldots$, $\Delta f_{\mathbf{h}}(X, \varphi_{\mathbf{h}} + Y)$ with coefficients $\omega_0, \omega_1, \ldots, \omega_{\mathbf{h}}$, respectively. From Proposition 4.9 (i) and (ii) it follows that the abscissa of the lowest vertex of the diagram $\Delta H(X, \varphi_{\mathbf{h}} + Y)$ equals

$$\omega_0 + \omega_1 \frac{\bar{b}_1}{b_0} + \cdots + \omega_{\mathbf{h}-1} \frac{\bar{b}_{\mathbf{h}-1}}{b_0} + \omega_{\mathbf{h}} \left(\frac{n_{\mathbf{h}-1}\bar{b}_{\mathbf{h}-1} - b_{\mathbf{h}-1}}{b_0} \right) . \tag{26}$$

The ordinate equals $\omega_{\mathbf{h}}$ hence we put $\omega_{\mathbf{h}} = \beta$. We want to choose $\omega_0, \omega_1, \ldots, \omega_{\mathbf{h}-1}$ in order to have

$$\omega_0 + \omega_1 \frac{\bar{b}_1}{b_0} + \cdots + \omega_{\mathbf{h}-1} \frac{\bar{b}_{\mathbf{h}-1}}{b_0} + \beta \left(\frac{n_{\mathbf{h}-1}\bar{b}_{\mathbf{h}-1} - b_{\mathbf{h}-1}}{b_0} \right) = \alpha \, . \tag{27}$$

Then

$$\omega_0 \bar{b}_0 + \omega_1 \bar{b}_1 + \cdots + \omega_{\mathbf{h}-1} \bar{b}_{\mathbf{h}-1} = \alpha b_0 - \beta(n_{\mathbf{h}-1}\bar{b}_{\mathbf{h}-1} - b_{\mathbf{h}-1}) \, . \tag{28}$$

Notice that αb_0 is an integer divisible by $n_{\mathbf{h}}$. The value of the right side is fixed. There are unknowns $\omega_0, \ldots, \omega_{\mathbf{h}-1}$ on the left side. We can apply the semigroup theory (e.g. [8]). Let us recall the notion of the conductor

$$c_k = (n_1 - 1)\bar{b}_1 + \cdots + (n_k - 1)\bar{b}_k - \bar{b}_0 + e_k, \quad (k = 1, \ldots, \mathbf{h}) \tag{29}$$

with the property that for every integer $c \ge c_k$ such that $c \equiv 0 (\operatorname{mod} e_k)$ there exists the unique sequence $\omega_0, \omega_1, \ldots, \omega_k$ such that $\omega_0 \ge 0$, $0 \le \omega_1 < n_1, \ldots, 0 \le \omega_k < n_k$ satisfying $c = \omega_0 \bar{b}_0 + \omega_1 \bar{b}_1 + \cdots + \omega_k \bar{b}_k$. Hence, it suffices to show that the right side R of (28) is greater than or equal to $c_{\mathbf{h}-1}$. Let us notice that the right side is divisible by $e_{\mathbf{h}-1} = n_{\mathbf{h}}$. It follows from the inequality $\beta \le n_{\mathbf{h}} - 1$ that

$$R = \alpha b_0 - \beta n_{\mathbf{h}-1}\bar{b}_{\mathbf{h}-1} + \beta b_{\mathbf{h}-1} \ge \alpha b_0 - (n_{\mathbf{h}} - 1) n_{\mathbf{h}-1}\bar{b}_{\mathbf{h}-1} + \beta b_{\mathbf{h}-1} \, . \tag{30}$$

Therefore

$$R \ge (\alpha n_{\mathbf{h}} - n_{\mathbf{h}} n_{\mathbf{h}-1}\bar{b}_{\mathbf{h}-1} + \beta b_{\mathbf{h}-1}) + n_{\mathbf{h}-1}\bar{b}_{\mathbf{h}-1} \, . \tag{31}$$

The number in parantheses in nonnegative. It follows from the fact that the chosen point B of the form (24) lies over the straight line $\pi_{\mathbf{h}-1}$ or on this line (Proposition 4.9 (iv)). Hence

$$R \geq n_{\mathbf{h}-1}\bar{b}_{\mathbf{h}-1}\ . \tag{32}$$

In order to show that $R \geq c_{\mathbf{h}-1}$ we study the difference $R - c_{\mathbf{h}-1}$. The first $\mathbf{h}-1$ components in the formula on $c_{\mathbf{h}-1}$ are written below in the opposite order:

$$\begin{aligned} R - c_{\mathbf{h}-1} &\geq n_{\mathbf{h}-1}\bar{b}_{\mathbf{h}-1} - (n_{\mathbf{h}-1}-1)\bar{b}_{\mathbf{h}-1} - (n_{\mathbf{h}-2}-1)\bar{b}_{\mathbf{h}-2} - \cdots - (n_1-1)\bar{b}_1 + \bar{b}_0 - e_{\mathbf{h}-1} \\ &\geq n_{\mathbf{h}-1}\bar{b}_{\mathbf{h}-1} - n_{\mathbf{h}-1}\bar{b}_{\mathbf{h}-1} + (\bar{b}_{\mathbf{h}-1} - n_{\mathbf{h}-2}\bar{b}_{\mathbf{h}-2}) + \cdots + (\bar{b}_2 - n_1\bar{b}_1) + \bar{b}_1 + \bar{b}_0 - e_{\mathbf{h}-1}\ . \end{aligned}$$

Since the numbers in parantheses are positive (God given inequality) we obtain

$$R - c_{\mathbf{h}-1} > \bar{b}_1 + \bar{b}_0 - e_{\mathbf{h}-1} \geq 0 \tag{33}$$

which finish the proof of the lemma. □

Main construction

Lemma 2 gives us some freedom to chose B. However, during the construction of the deformation $F_t = f + tH$ the point B is unique (in the fixed coordinate system). Every characteristic exponent may be written in the form

$$\frac{b_k}{b_0} = \frac{m_k}{n_1 \dots n_k}\ , \quad \mathrm{GCD}(n_k, m_k) = 1\ , \quad k = 1, \dots, \mathbf{h}\ . \tag{34}$$

The pairs (n_1, m_1), …, $(n_{\mathbf{h}}, m_{\mathbf{h}})$ are called the characteristic Puiseux pairs. Applying the Euclid algorithm to the last characteristic pair we choose the unique integers i, j such that

$$\begin{cases} m_{\mathbf{h}}j - n_{\mathbf{h}}i = 1 \\ 0 < i < m_{\mathbf{h}} \\ 0 < j < n_{\mathbf{h}} \end{cases}\ . \tag{35}$$

Then we put

$$\tilde{\alpha} := \frac{\bar{b}_{\mathbf{h}} - b_{\mathbf{h}} + i}{N_{\mathbf{h}-1}}\ , \qquad \tilde{\beta} := n_{\mathbf{h}} - j\ . \tag{36}$$

We choose by Lemma 4.10 $\omega_0, \omega_1, \dots, \omega_{\mathbf{h}} = \tilde{\beta}$ such that the lower vertex of the diagram $\Delta H(X, y_{\mathbf{h}} + Y)$ (25) equals $B(\tilde{\alpha}, \tilde{\beta})$. Recall that

$$0 \leq \omega_1 < n_1\ ,\ \dots\ ,\ 0 \leq \omega_{\mathbf{h}} < n_{\mathbf{h}}\ . \tag{37}$$

Now, we want to finish the proof. In the begining of this section we discussed two cases that allows to compare $T_*(f, X)$ and $T_*(F_t, X)$. Without loss of generality we assume that $b_0 = v_0$.

Proposition 4.11. (first case) *If one of the following conditions holds:*

(a) $\ell_{\mathbf{h}} = 1$,
(b) $\ell_{\mathbf{h}} \geq 2$ *and* $(b_{\mathbf{h}} - v_{\ell_{\mathbf{h}}-1})(n_{\mathbf{h}} - \tilde{\beta}) > 1$,
(c) $\ell_{\mathbf{h}} \geq 2$ *and* $(b_{\mathbf{h}} - v_{\ell_{\mathbf{h}}-1})(n_{\mathbf{h}} - \tilde{\beta}) = 1$ *but* $n_{\mathbf{h}} \geq 3$

then

(i) $T_*(F_t, X) = T_*(f, X) = \operatorname{cycle}(\varphi_1) \cup \cdots \cup \operatorname{cycle}(\varphi_{\ell_{\mathbf{h}}})$.
(ii) *If* $\ell < \ell_{\mathbf{h}}$ *then* $\hat{\mu}(F_t, \varphi_\ell) = \hat{\mu}(f, \varphi_\ell)$.
(iii) $\hat{\mu}(F_t, \varphi_{\ell_{\mathbf{h}}}) = \hat{\mu}(f, \varphi_{\ell_{\mathbf{h}}}) - \frac{1}{N_{\mathbf{h}-1}}$.

Proposition 4.12. (second case)
If $\ell_{\mathbf{h}} \geq 2$ *and* $(b_{\mathbf{h}} - v_{\ell_{\mathbf{h}}-1})(n_{\mathbf{h}} - \beta) = 1$ *and* $n_h = 2$ *then*

(i) $T_*(F_t, X) = \operatorname{cycle}(\varphi_1) \cup \cdots \cup \operatorname{cycle}(\varphi_{\ell_{\mathbf{h}}-1})$.
(ii) *If* $\ell < \ell_{\mathbf{h}}$ *then* $\hat{\mu}(F_t, \varphi_\ell) = \hat{\mu}(f, \varphi_\ell)$.
(iii) $\hat{\mu}(f, \varphi_{\ell_{\mathbf{h}}}) = \frac{1}{N_{\mathbf{h}-1}}$.

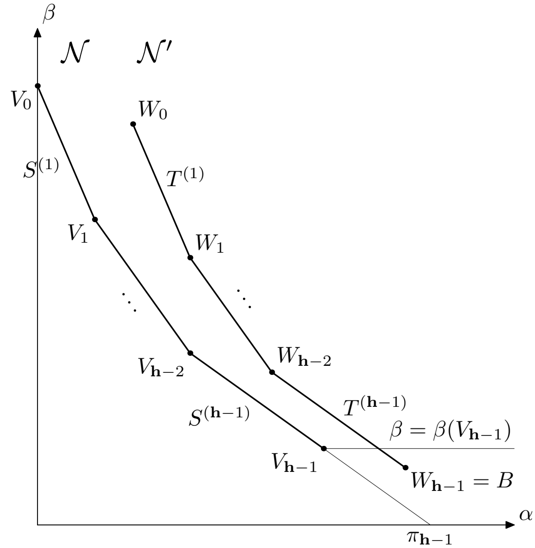
To finish the proof it suffices to verify propositions. Before this we should study relation of the Newton polygons of the diagrams $\Delta f(X, y_{\mathbf{h}}+Y)$ and $\Delta H(X, y_{\mathbf{h}}+Y)$. It follows from Corollary 4.7 that the Newton polygon of the first diagram has $\mathbf{h}$ segments $S^{(1)}, \ldots, S^{(\mathbf{h})}$ with respective inclinations $\frac{b_1}{b_0}, \ldots, \frac{b_{\mathbf{h}}}{b_0}$. Let $V_0, V_1, \ldots V_{\mathbf{h}}$ be succesive vertices of the first diagram (ordered from up to down). We have $S_k = \overline{V_{k-1}V_k}$ $(k = 1, \ldots, \mathbf{h})$; we use bar to denote segment. From Corollary 4.8 and from (25) we conclude that the Newton polygon of the second diagram has $\mathbf{h}-1$ segments $T^{(1)}, \ldots, T^{(\mathbf{h}-1)}$ with respective inclinations $\frac{b_1}{b_0}, \ldots, \frac{b_{\mathbf{h}-1}}{b_0}$. Let $W_0, W_1, \ldots W_{\mathbf{h}-1}$ be succesive vertices of the second diagram. We have $T_k = \overline{V_{k-1}V_k}$ $(k = 1, \ldots, \mathbf{h}-1)$. Recall that $W_{\mathbf{h}-1} = B(\tilde{\alpha}, \tilde{\beta})$ from the construction in Lemma 4.10 For a point (vertex) V we will write $\alpha(V)$ (resp. $\beta(V)$) to denote its abscissa (resp. ordinate).

Proposition 4.13.

(I) *For two above Newton polygons we consider the sets of first* $\mathbf{h}-1$ *segments* $\mathcal{N} = \{S^{(1)}, \ldots, S^{(\mathbf{h}-1)}\}$ *and* $\mathcal{N}' = \{T^{(1)}, \ldots, T^{(\mathbf{h}-1)}\}$. *We claim that* $\mathcal{N}'$ *lies over* $\mathcal{N}$ *in the weak sense: only the last segment of* $\mathcal{N}'$ *and the last segment of* $\mathcal{N}$ *may lay on the same straigh line.*
(II) *The inclination of straight line determined by* $V_{\mathbf{h}-1}$ *and* $W_{\mathbf{h}-1}$ *equals*

$$\frac{\alpha(W_{\mathbf{h}-1}) - \alpha(V_{\mathbf{h}-1})}{\beta(V_{\mathbf{h}-1}) - \beta(W_{\mathbf{h}-1})} = \frac{b_{\mathbf{h}}}{b_0} - \frac{1}{b_0(n_{\mathbf{h}} - \tilde{\beta})}\,.$$

Proof. (I). The vertices $V_{\mathbf{h}-2}$ and $V_{\mathbf{h}-1}$ determines the straight line $\pi_{\mathbf{h}-1}$. From the construction of $W_{\mathbf{h}-1} = B$ we have $\beta(W_{\mathbf{h}-1}) < \beta(V_{\mathbf{h}-1}) = n_{\mathbf{h}}$ and $W_{\mathbf{h}-1}$ lies over the line $\pi_{\mathbf{h}-1}$ or on this line.



Taking into consideration a geometrical argument to finish the proof if suffices to show that

$$|T^{(k)}| \leq e_{k-1} - e_k = |S^{(k)}| \text{ for } k = 1, \ldots, \mathbf{h} - 1 \,. \tag{38}$$

By Corollaries 4.8 and 4.7 to each diagram $\Delta f_1(X, y_{\mathbf{h}} + Y), \ldots, \Delta f_{\mathbf{h}}(X, y_{\mathbf{h}} + Y)$, $\Delta f(X, y_{\mathbf{h}} + Y)$ we assign the succesive inclinations that appear in their Newton polygons. We write ∞ if a diagram does not touch the horizontal axis. We write the multiplicities in the meaning of Theorem 2.1 (i) under the values.

$$\begin{array}{lllllll}
\Delta f_1(X, y_{\mathbf{h}} + Y) & \underbrace{(b_1/b_0)}_{1} & & & & & \\
\Delta f_2(X, y_{\mathbf{h}} + Y) & \underbrace{(b_1/b_0)}_{(n_1-1)} & \underbrace{(b_2/b_0)}_{1} & & & & \\
\Delta f_3(X, y_{\mathbf{h}} + Y) & \underbrace{(b_1/b_0)}_{(n_1-1)n_2} & \underbrace{(b_2/b_0)}_{(n_2-1)} & \underbrace{(b_3/b_0)}_{1} & & & \\
\cdots & \cdots & \cdots & \cdots & \cdots & & \\
\Delta f_{\mathbf{h}}(X, y_{\mathbf{h}} + Y) & \underbrace{(b_1/b_0)}_{(n_1-1)n_2 \ldots n_{\mathbf{h}}} & \underbrace{(b_2/b_0)}_{(n_2-1)n_3 \ldots n_{\mathbf{h}}} & \underbrace{(b_3/b_0)}_{(n_3-1)n_4 \ldots n_{\mathbf{h}}} & \cdots & \underbrace{(b_{\mathbf{h}-1}/b_0)}_{(n_{\mathbf{h}-1}-1)} & \underbrace{\infty}_{1} \\
\Delta f(X, y_{\mathbf{h}} + Y) & \underbrace{(b_1/b_0)}_{e_0-e_1} & \underbrace{(b_2/b_0)}_{e_1-e_2} & \underbrace{(b_3/b_0)}_{e_2-e_3} & \cdots & \underbrace{(b_{\mathbf{h}-1}/b_0)}_{e_{\mathbf{h}-2}-e_{\mathbf{h}-1}} & \underbrace{(b_{\mathbf{h}}/b_0)}_{e_{\mathbf{h}-1}=n_{\mathbf{h}}}
\end{array}$$

Applying (37) we can estimate

$$
\begin{aligned}
|T^{(1)}|_{\mathbf{V}} &= 1\cdot\omega_1+(n_1-1)\cdot\omega_2+(n_1-1)n_2\cdot\omega_3+\cdots+(n_1-1)n_2\dots n_{\mathbf{h}-1}\cdot\omega_{\mathbf{h}}\\
&\le (n_1-1)+(n_1-1)(n_2-1)+(n_1-1)n_2(n_3-1)+\cdots+(n_1-1)n_2\dots n_{\mathbf{h}-1}(n_{\mathbf{h}}-1)\\
&= (n_1-1)n_2\dots n_{\mathbf{h}}=e_0-e_1=|S^{(1)}|_{\mathbf{V}}
\end{aligned}
$$

We reason analogously for other segments and we finish the proof of (I).

Now, we prove (II). From Proposition 4.9 (iii) we obtain the equation of the straight line $\pi_{\mathbf{h}}$ that contains the segment $S_{\mathbf{h}}=\overline{V_{\mathbf{h}-1}V_{\mathbf{h}}}$:

$$\alpha b_0+\beta b_{\mathbf{h}}=\bar{b}_{\mathbf{h}}n_{\mathbf{h}}\ . \tag{39}$$

From (35) and (36) we obtain that the coordinates of the point $W_{\mathbf{h}-1}=B$ satisfy:

$$\tilde{\alpha}b_0+\tilde{\beta}b_{\mathbf{h}}=\bar{b}_{\mathbf{h}}n_{\mathbf{h}}-1\ . \tag{40}$$

Substituting $\beta=n_{\mathbf{h}}$ to (39) we obtain

$$\alpha(V_{\mathbf{h}-1})=\frac{n_{\mathbf{h}}(\bar{b}_{\mathbf{h}}-b_{\mathbf{h}})}{b_0}\ . \tag{41}$$

By (40) we have

$$\frac{\alpha(W_{\mathbf{h}-1})-\alpha(V_{\mathbf{h}-1})}{\beta(V_{\mathbf{h}-1})-\beta(W_{\mathbf{h}-1})}=\frac{\tilde{\alpha}b_0-n_{\mathbf{h}}\bar{b}_{\mathbf{h}}+n_{\mathbf{h}}b_{\mathbf{h}}}{(n_{\mathbf{h}}-\tilde{\beta})b_0}=\frac{b_{\mathbf{h}}}{b_0}-\frac{1}{b_0(n_{\mathbf{h}}-\tilde{\beta})}\ . \tag{42}$$

This finishes the proof. □

In the next proposition we study relations between the diagrams $\Delta f(X,\varphi_\ell+Y)$ and $\Delta H(X,\varphi_\ell+Y)$ for $\ell<\ell_{\mathbf{h}}$. By using the Teissier notation of the diagram

$$\Delta=\sum_{i=1}^{n}\left\{\frac{a_i}{b_i}\right\}\ , a_i,b_i>0,\ \text{at least one of } a_i,b_i \text{ is finite}\ ,\quad i=1,\dots,n\ , \tag{43}$$

we can assign inclinations directly to the diagram. To $\{\frac{a_i}{b_i}\}$ we assign $\frac{a_i}{b_i}$ with convetion $\frac{a_i}{\infty}=0$ and $\frac{\infty}{b_i}=\infty$ $(i=1,\dots,n)$. For a rational $\theta>0$ we define the transformation $[\Delta]_\theta$ of the diagram Δ which replace the components with inclination strictly geater then θ by the respective components with inclination θ. We write

$$[\Delta]_\theta=\sum_{\frac{a_i}{b_i}\le\theta}\left\{\frac{a_i}{b_i}\right\}+\sum_{\frac{a_i}{b_i}>\theta}\left\{\frac{\theta b_i}{b_i}\right\}\ . \tag{44}$$

Clearly

$$\text{if } \Delta\subset\Delta' \text{ then } [\Delta]_\theta\subset[\Delta']_\theta\ . \tag{45}$$

Recall that $\theta_\ell=v_\ell/v_0$ (1). Clearly, $\theta_\ell=o_f(\varphi_\ell)$ (22).

Proposition 4.14. *Let* $\ell < \ell_{\mathbf{h}}$. *Then*

(a) $\Delta f(X, \varphi_\ell + Y) = [\Delta f(X, y_{\mathbf{h}} + Y)]_{\theta_\ell}$,
(b) $\Delta H(X, \varphi_\ell + Y) \subset [\Delta H(X, y_{\mathbf{h}} + Y)]_{\theta_\ell}$.

Proof. We use methods of Lemma 7.1 from [20]. □

We can consider the diagrams generated by points $V_1, \ldots, V_n$ in $\mathbb{R}^2_+$. By $\Delta\{V_1, \ldots, V_n\}$ we mean the convex hull of the union $V_1 + \mathbb{R}^2_+ \cup \cdots \cup V_n + \mathbb{R}^2_+$. Recall that $F_t = f + tH$.

Proposition 4.15. *Let* $\ell < \ell_{\mathbf{h}}$. *Then*

(i) $\Delta f(X, \varphi_\ell + Y) = \Delta F_t(X, \varphi_\ell + Y)$,
(ii) $\mathrm{cycle}(\varphi_\ell) \subset T_*(F_t, X)$.
(iii) $\hat{\mu}(f, \varphi_\ell) = \hat{\mu}(F_t, \varphi_\ell)$.

Proof. (i) The line ρ_ℓ with inclination θ_ℓ supporting $\Delta f(X, y_{\mathbf{h}} + Y)$ crosses the horizontal axis at the point A_ℓ. We have

$$[\Delta f(X, y_{\mathbf{h}} + Y)]_{\theta_\ell} = \Delta\{V_0, \ldots, V_{k-1}, A_\ell\} \tag{46}$$

with the smallest k such that $\theta_\ell \leq \frac{b_k}{b_0}$. Analogously, the line ρ'_ℓ with the same inclination supporting $\Delta H(X, y_{\mathbf{h}} + Y)$ meets the horizontal axis at B_ℓ. We have

$$[\Delta H(X, y_{\mathbf{h}} + Y)]_{\theta_\ell} = \Delta\{W_0, \ldots, W_{k-1}, B_\ell\} \tag{47}$$

where k is the smallest with $\theta_\ell \leq \frac{b_k}{b_0}$. Both parts of Proposition 4.13 give

$$\Delta\{W_0, \ldots, W_{k-1}, B_\ell\} \subset \Delta\{V_0, \ldots, V_{k-1}, A_\ell\} \, . \tag{48}$$

Hence $[\Delta H(X, y_{\mathbf{h}} + Y)]_{\theta_\ell} \subset [\Delta f(X, y_{\mathbf{h}} + Y)]_{\theta_\ell}$. From Proposition 4.14 we obtain $\Delta H(X, \varphi_\ell + Y) \subset \Delta f(X, \varphi_\ell + Y)$. This gives (i) for sufficiently small $t \neq 0$. Both parts (ii) and (iii) follow from (i). □

Below, we proof a useful lemma. Let us recall a classical fact.

Property 4.16. *Nonzero polynomials* $f, g \in \mathbb{C}[X]$ *have a common root if and only if there exist nonzero polynomials* $a, b \in \mathbb{C}[X]$, $\deg a < \deg g$, $\deg b < \deg f$ *such that* $af - bg = 0$ *in* $\mathbb{C}[X]$.

Lemma 4.17. *Let* $f, g \in \mathbb{C}[X]$ *be polynomials without common roots. Then for small* $t \neq 0$ $f + tg \in \mathbb{C}[X]$ *has only single roots.*

Proof. Let $h = fg' - f'g \in \mathbb{C}[X]$. From property we conclude that h is nonzero polynomial. Let

$$Z = \left\{ -\frac{f(c)}{g(c)} \, : \, h(c) = 0 \text{ and } g(c) \neq 0 \right\} . \tag{49}$$

Clearly, Z is finite (may be empty). We will show that for

$$t \in \mathbb{C} \setminus (Z \cup \{0\}) \tag{50}$$

the polynomial F_t has only single roots. For the contrary let us assume that $F_t(c) = F_t'(c) = 0$. Hence

$$\begin{cases} f(c) + t\,g(c) &= 0 \\ f'(c) + t\,g'(c) &= 0 \end{cases} . \tag{51}$$

Since the system has nonzero solution $(1, t)$, the determinant must be zero. Hence $h(c) = 0$. It must be $g(c) \neq 0$. From the first equation we obtain $t = -\frac{f(c)}{g(c)}$ which contadicts (50). □

Remark 4.18. (e.g.[20]) It is covenient to apply the initial form defined by the pair of positive weights (a, b). For $f = \sum c_{\alpha\beta} X^\alpha Y^\beta \in \mathbb{C}\{X^*, Y\}$ we put $\mathrm{ord}_{(a,b)} f = \min\{a\alpha + b\beta\ :\ c_{\alpha,\beta} \neq 0\}$, $\mathrm{in}_{(a,b)} f = \sum c_{\alpha\beta} X^\alpha Y^\beta$ where (α, β) correspond to nonzero coefficients and $a\alpha + b\beta = \mathrm{ord}_{\mathbf{v}} f$. We put $\mathrm{ord}_{\mathbf{v}} 0 = \infty$ and $\mathrm{in}_{\mathbf{v}} 0 = 0$. For $f, g \in \mathbb{C}\{X^*, Y\}$ we have $\mathrm{ord}_{\mathbf{v}}(fg) = \mathrm{ord}_{\mathbf{v}} f + \mathrm{ord}_{\mathbf{v}} g$ and $\mathrm{in}_{\mathbf{v}}(fg) = (\mathrm{in}_{\mathbf{v}} f)(\mathrm{in}_{\mathbf{v}} g)$.

Verification of Propositions 4.11 and 4.12

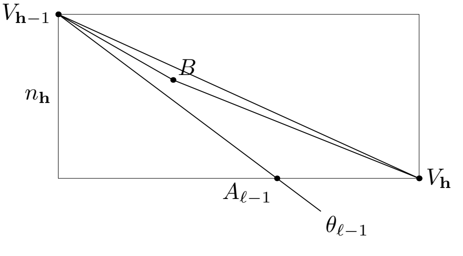
Below, we will check both propositions.

Proof. Case (I) (a). The equality $\ell_{\mathbf{h}} = 1$ means that the series y has one characteristic pair and that the first exponent $\frac{v_1}{v_0} = \frac{b_1}{b_0} = \frac{m_1}{n_1}$ is characteristic. In this case $\Delta f(X, Y)$ has one segment S which joins $(0, n_1)$ and $(m_1, 0)$. By Properties 4.4 and 4.5 we have (up to a nonzero constant)

$$\mathrm{in}(f, S) = Y^{n_1} - a_1^{n_1} X^{m_1} .$$

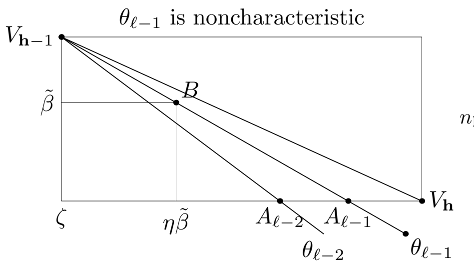
We put $F_t = f + tX^{\tilde{\alpha}} Y^{\tilde{\beta}}$ and we reason as in the Bodin's case. We have $T_*(f, X) = T_*(F_f, X) = \{0\}$ and $\hat{\mu}(F_t, 0) = \hat{\mu}(f, 0) - 1$.

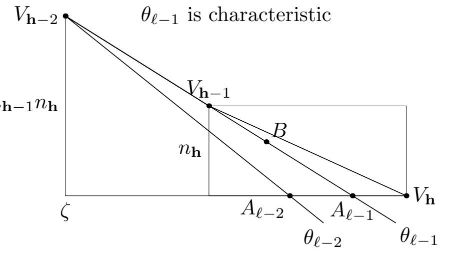
Case (I) (b). Now, we assume that $\ell_{\mathbf{h}} \geq 2$ and $(b_{\mathbf{h}} - v_{\ell_{\mathbf{h}}-1})(n_{\mathbf{h}} - \beta) > 1$. For simplicity we write $\ell = \ell_{\mathbf{h}}$. We have $\deg \varphi_\ell = \theta_{\ell-1}$. Since $|\mathcal{N}(F_t, \varphi_\ell)| > 1$ then $\varphi_\ell \in T_*(F_t, X)$.



Hence $\mathrm{cycle}(\varphi_1) \cup \cdots \cup \mathrm{cycle}(\varphi_{\ell_{\mathbf{h}}}) \subset T_*(F_t, X)$. We obtain the opposite inclusion from the equality $(f, X)_0 = (F_t, X)_0$ (which follows from Proposition 4.13) and by counting solutions of $F_t = 0$. Part (ii) follows from Proposition 4.15 (iii). As the case of Bodin we check that $\hat{\mu}(F_t, \varphi_\ell) = \hat{\mu}(f, \varphi_\ell) - \frac{1}{N_{\mathbf{h}-1}}$.

Case (I) (c) and Case (II). Let us assume that $\ell_{\mathbf{h}} \geq 2$ and $(b_{\mathbf{h}} - v_{\ell_{\mathbf{h}}-1})(n_{\mathbf{h}} - \beta) = 1$. As earlier we write $\ell = \ell_{\mathbf{h}}$. By using notation of Proposition 4.13 and from the proof of Proposition 4.15 the segments $V_{\mathbf{h}-1}A_{\ell-1}$ and $W_{\mathbf{h}-1}B_{\ell-1}$ lay on the same straigh line with the inclination $\theta_{\ell-1}$ ($W_{\mathbf{h}-1} = B$ and $A_{\ell-1} = B_{\ell-1}$).





We study $F_t(X, \varphi_{\ell-1} + Y)$. We have $\deg \varphi_{\ell-1} = \theta_{\ell-2}$ (we put $\theta_0 = 0$). Applying Property 4.5 we compute

$$\mathrm{in}_{(1,\theta_{\ell-1})} F_t(X, \varphi_{\ell-1} + Y) = \mathrm{in}_{(1,\theta_{\ell-1})} f(X, \varphi_{\ell-1} + Y) + t\, \mathrm{in}_{(1,\theta_{\ell-1})} H(X, \varphi_{\ell-1} + Y)\ .$$

Let us denote this form by I. For nonzero c, d and nonnegative ζ, η we have

$$\begin{aligned} I &= cX^{\zeta}\left(Y^{u_{\ell-1}} - a_{\ell-1}^{u_{\ell-1}} X^{\theta_{\ell-1}u_{\ell-1}}\right)^{n_{\mathbf{h}}} + t\, d^{\tilde{\beta}} X^{\eta\tilde{\beta}} \left(Y^{u_{\ell-1}} - a_{\ell-1}^{u_{\ell-1}} X^{\theta_{\ell-1}u_{\ell-1}}\right)^{\tilde{\beta}} \\ &= cX^{\zeta}\left(Y^{u_{\ell-1}} - a_{\ell-1}^{u_{\ell-1}} X^{\theta_{\ell-1}u_{\ell-1}}\right)^{\tilde{\beta}} \left[\left(Y^{u_{\ell-1}} - a_{\ell-1}^{u_{\ell-1}} X^{\theta_{\ell-1}u_{\ell-1}}\right)^{n_{\mathbf{h}}-\tilde{\beta}} + \frac{td^{\tilde{\beta}}}{c} X^{\zeta-\eta\tilde{\beta}}\right]\ . \end{aligned}$$

The right factor is nodegenerate by Lemma 4.17. If $\tilde{\beta} > 1$ (case (I) c) then $a_{\ell-1}X^{\theta_{\ell-1}}$ is a multiple root of $\mathrm{in}_{(1,\theta_{\ell-1})} F_t$. Hence

$$\varphi_{\ell} = \varphi_{\ell-1} + a_{\ell-1}X^{\theta_{\ell-1}} \in T_*(F_t, X)\ . \tag{52}$$

To obtain (ii) we reason as in the previous case. We have

$$\begin{aligned} \hat{\mu}(f, \varphi_{\ell}) &= 2\,\mathrm{Area}(V_{\mathbf{h}-1}A_{\ell-1}V_{\mathbf{h}}) - |A_{\ell-1}V_{\mathbf{h}}|\ , \\ \hat{\mu}(F_t, \varphi_{\ell}) &= 2\,\mathrm{Area}(BA_{\ell-1}V_{\mathbf{h}}) - |A_{\ell-1}V_{\mathbf{h}}| \end{aligned}$$

where $|\dots|$ stands for the length of a segment. Clearly $\hat{\mu}(F_t, \varphi_{\ell}) = \hat{\mu}(f, \varphi_{\ell}) - \frac{1}{N_{\mathbf{h}}}$.

When $\tilde{\beta} = 1$ (equivalently $n_{\mathbf{h}} = 2$, case (II)) the form I is nondegenerate. Hence $T_*(f, X) = \mathrm{cycle}(\varphi_1) \cup \dots \cup \mathrm{cycle}(\varphi_{\ell_{\mathbf{h}}-1})$. We obtain $\hat{\mu}(f, \varphi_{\ell}) = \frac{1}{N_{\mathbf{h}}}$. This finishes the proof of Propositions 4.11 and 4.12 and the proof of Theorem 1.1 □

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