Analytic and Algebraic Geometry 4
Łódź University Press 2022, 61–66
DOI: https://doi.org/10.18778/8331-092-3.05

ON LÊ'S FORMULA IN ARBITRARY CHARACTERISTIC

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STRESZCZENIE. In this note we extend, to arbitrary characteristic, Lê's formula (*Calculation of Milnor number of isolated singularity of complete intersection.* Funct. Anal. Appl. 8 (1974), 127–131).

1. Introduction

Let K be an algebraically closed field of characteristic $p \geqslant 0$. For any power series $f, g \in K[[x,y]]$ we put $i_0(f,g) := \dim_K K[[x,y]]/(f,g)$ and call it the *intersection multiplicity* of f and g. We denote by $[f,g]$ the Jacobian determinant of (f,g), that is $[f,g] = \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}$.

For any formal power series $f \in K[[x,y]]$ we denote by $\mathrm{ord} f$ the *order* of f. Any power series of order one is called a *regular parameter*.

Let $f \in K[[x,y]]$ be a power series without constant term. The *Milnor number* of $f \in K[[x,y]]$ is $\mu(f) := i_0\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$. Suppose that f is reduced, that is, it has no multiple factors. We put $\mathcal{O}_f = K[[x,y]]/(f)$, $\overline{\mathcal{O}_f}$ the integral closure of $\mathcal{O}_f$ in the total quotient ring of $\mathcal{O}_f$. Let $\mathcal{C}$ be the *conductor* of $\mathcal{O}_f$, that is the largest ideal in $\mathcal{O}_f$ which remains an ideal in $\overline{\mathcal{O}_f}$. We define $c(f) = \dim_K \overline{\mathcal{O}}_f/\mathcal{C}$ (the *degree of conductor*) and $r(f)$ the number of irreducible factors of f.

We define

$$\overline{\mu}(f) := c(f) - r(f) + 1.$$

If the characteristic of K is zero then $\mu(f) = \overline{\mu}(f)$. See [GB-Pł, Proposition 2.1] for other properties of $\overline{\mu}(f)$.

2010 *Mathematics Subject Classification.* Primary 14H20, Secondary 32S05.

Key words and phrases. Milnor number, jacobian determinant.

The first-named author was partially supported by the Spanish Project PID2019-105896GB-I00 funded by MCIN/AEI/10.13039/501100011033.

The main result of this note is to extend Lê's formula (see [L] and [G]) to arbitrary characteristic:

Theorem A (Lê's formula in arbitrary characteristic)**.** *Let l be a regular parameter. Let $f, g \in K[[x,y]]\backslash\{0\}$ be coprime without constant term. Suppose that f is reduced and let $f = f_1 \cdots f_r$ be the factorization of f into irreducible factors. If $i_0(f_i, l) \not\equiv 0 \pmod p$ for $i = 1, \dots, r$ then*

$$i_0(f, [f,g]) \geqslant \overline{\mu}(f) + i_0(f,g) - 1. \tag{1}$$

The equality in (1) *holds if and only if $i_0(f_i, g) \not\equiv 0 \pmod p$ for $i = 1, \dots, r$.*

Corollary to Theorem A. *If f is irreducible and* $\mathrm{ord} f \not\equiv 0 \pmod p$ *then*

$$i_0(f, [f,g]) \geqslant c(f) + i_0(f,g) - 1. \tag{2}$$

The equality in (2) *holds if and only if $i_0(f,g) \not\equiv 0 \pmod p$.*

Remark 1.1. *The assumption* $\mathrm{ord} f \not\equiv 0 \pmod p$ *in the above corollary is irrelevant (see* [H-R-S1, Corollary 2.4]*).*

2. Proof of Lê's formula

Let t be a variable. A *parametrization* is a pair $(x(t), y(t)) \in K[[t]]^2\backslash\{(0,0)\}$ such that $x(0) = y(0) = 0$. We say that the parametrization $(x(t), y(t))$ is *good* if the field of fractions of the ring $K[[x(t), y(t)]]$ is equal to the field $K((t))$. By the Normalization Theorem (see for example [Pł, Theorem 2.1]), any irreducible power series in $K[[x,y]]$ admits a good parametrization.

The proof of Lê's formula will be a consequence of two lemmas. Let $f, g \in K[[x,y]]\backslash\{0\}$ be without constant term.

Lemma 2.1 (Teissier's lemma in arbitrary characteristic)**.** *Let l be a regular parameter and let $f \in K[[x,y]]$ be a reduced power series with factorization $f = f_1 \cdots f_r$ into irreducible factors. Suppose that $i_0(f_i, l) \not\equiv 0 \pmod p$ for $i = 1, \dots, r$. Then*

$$i_0(f, [f,l]) = \overline{\mu}(f) + i_0(f,l) - 1.$$

Proof. See [GB-Pł, Proposition 2.1(iii)]. □

The following lemma generalizes to arbitrary characteristic *Delgado's Formula* (see [D, Proposition 2.1.1] or [Ca, Proposition 2.4.1]).

Lemma 2.2 (Delgado's formula)**.** *Let $f, g \in K[[x,y]]\backslash\{0\}$ be coprime and l be a regular parameter. Suppose that f is reduced and $f = f_1 \cdots f_r$ is its factorization into irreducible factors. If $i_0(f_i, l) \not\equiv 0 \pmod p$ for $i = 1, \dots, r$ then*

$$i_0(f, [f,g]) \geqslant i_0(f,g) + i_0(f,[f,l]) - i_0(f,l) \tag{3}$$

with equality if and only if $i_0(f_i, g) \not\equiv 0 \pmod p$ for any irreducible factor f_i of f.

Proof. We may assume, without loss of generality, that $l(x,y)=x$, hence $[f,l]=-\frac{\partial f}{\partial y}$ and $i_0(f,l)=\mathrm{ord} f(0,y)$. Let f_i be an irreducible factor of f and let $\gamma(t)=(x(t),y(t))$ be a good parametrization of the curve $\{f_i(x,y)=0\}$. We get

$$\frac{\partial f}{\partial x}(\gamma(t))x'(t)+\frac{\partial f}{\partial y}(\gamma(t))y'(t)=0. \tag{4}$$

Since f and g are coprime, f_i is not a factor of g, that is $g(x(t),y(t))\neq 0$ and

$$\frac{\partial g}{\partial x}(\gamma(t))x'(t)+\frac{\partial g}{\partial y}(\gamma(t))y'(t)=\frac{d}{dt}g(\gamma(t)). \tag{5}$$

Consider the system, in the unknowns U and V:

$$\begin{cases} \frac{\partial f}{\partial x}(\gamma(t))U+\frac{\partial f}{\partial y}(\gamma(t))V=0 \\ \frac{\partial g}{\partial x}(\gamma(t))U+\frac{\partial g}{\partial y}(\gamma(t))V=\frac{d}{dt}g(\gamma(t)). \end{cases} \tag{6}$$

By (4) and (5) the pair $(x'(t),y'(t))$ is a solution of the system (6). By Cramer's identities we get $[f,g](\gamma(t))x'(t)=-\frac{\partial f}{\partial y}(\gamma(t))\frac{d}{dt}g(\gamma(t))$ and taking orders we obtain

$$\mathrm{ord}[f,g](\gamma(t))+\mathrm{ord}x'(t)=\mathrm{ord}\frac{\partial f}{\partial y}(\gamma(t))+\mathrm{ord}\frac{d}{dt}g(\gamma(t)). \tag{7}$$

Since $i_0(f_i,x)\not\equiv 0 \pmod p$ we have $\mathrm{ord}\, x(t)\not\equiv 0 \pmod p$ and consequently $\mathrm{ord}\, x'(t)=\mathrm{ord}\, x(t)-1$. Analogously $\mathrm{ord}\frac{d}{dt}g(\gamma(t))\geqslant \mathrm{ord}g(\gamma(t))-1$, with equality if and only if, $\mathrm{ord}g(\gamma(t))=i_0(f_i,g)\not\equiv 0 \pmod p$. From (7) we get

$$i_0(f_i,[f,g])+i_0(f_i,x)\geqslant i_0(f_i,g)+i_0\left(f_i,\frac{\partial f}{\partial y}\right), \tag{8}$$

with equality if $i_0(f_i,g)\not\equiv 0 \pmod p$. Summing up the inequalities (8), we obtain

$$i_0(f,[f,g])+i_0(f,x)\geqslant i_0(f,g)+i_0\left(f,\frac{\partial f}{\partial y}\right)$$

with equality if $i_0(f_i,g)\not\equiv 0 \pmod p$, for $i=1,\dots,r$. □

Remark 2.3. *A particular case of Delgado's formula in arbitrary characteristic appears in* [H-R-S2, Lemma 3.5].

Proof of Theorem A It is a consequence of Lemmas 2.1 and 2.2. □

3. The case of characteristic zero

If the characteristic of K is zero then we have the following version of Lê's formula.

Theorem B (Lê's formula in zero characteristic)**.** *Let $f,g\in K[[x,y]]\backslash\{0\}$ be without constant term. Then*

$$i_0(f,[f,g])=\mu(f)+i_0(f,g)-1. \tag{9}$$

The left-hand side of (9) *is infinite if and only if the right-hand side is so.*

The following lemma is well-known (see for example [CN-Pł]):

Lemma 3.1. *Let $f, g \in K[[x,y]]\backslash\{0\}$ be without constant term.*

(1) $i_0(f,g) = +\infty$ *if and only if* f *and* g *are not coprime.*
(2) $\mu(f) = +\infty$ *if and only if* f *is not reduced.*

The following general property also will be useful:

Property 3.2. *Let $h(x,y) \in K[[x,y]]$ be an irreducible power series. Let $\gamma(t) = (x(t), y(t))$ be a good parametrization of $h(x,y)$. Then $\frac{\partial h}{\partial x}(\gamma(t)) \neq 0$ or $\frac{\partial h}{\partial y}(\gamma(t)) \neq 0$.*

Proof. Suppose that $\frac{\partial h}{\partial x}(\gamma(t)) = 0$ and $\frac{\partial h}{\partial y}(\gamma(t)) = 0$. This implies $\frac{\partial h}{\partial x} \equiv 0 \pmod{h}$ or $\frac{\partial h}{\partial y} \equiv 0 \pmod{h}$. Hence $\operatorname{ord}\frac{\partial h}{\partial x} \geqslant \operatorname{ord} h$ and $\operatorname{ord}\frac{\partial h}{\partial y} \geqslant \operatorname{ord} h$. This is a contradiction since if the characteristic of K is zero then $\operatorname{ord}\frac{\partial h}{\partial x} = \operatorname{ord} h - 1$ or $\operatorname{ord}\frac{\partial h}{\partial y} = \operatorname{ord} h - 1$. □

Proof of Theorem B If $\mu(f) + i_0(f,g)$ is finite then $\mu(f)$ is also. Hence, by the second part of Lemma 3.1, f is reduced and $\mu(f) = \overline{\mu}(f)$. Therefore, in this case, Theorem B follows from Theorem A.

The case where one of the two sides of (9) is infinite follows from the following proposition, which is equivalent to [Sz, Theorem 3.6].

Proposition 3.3. *Let $f, g \in K[[x,y]]\backslash\{0\}$ be without constant term. The following two conditions are equivalent:*

(1) $\mu(f) = +\infty$ *or* $i_0(f,g) = +\infty$.
(2) $i_0(f, [f,g]) = +\infty$.

Proof. Suppose that $\mu(f) = +\infty$ or $i_0(f,g) = +\infty$. In order to prove the equality $i_0(f,[f,g]) = +\infty$, we distinguish two cases.

Case 1: $\mu(f) = +\infty$. There is an irreducible power series $h \in K[[x,y]]$ such that $f \equiv 0 \pmod{h^2}$. Therefore $\frac{\partial f}{\partial x} \equiv 0 \pmod{h}$ and $\frac{\partial f}{\partial y} \equiv 0 \pmod{h}$ which implies $[f,g] \equiv 0 \pmod{h}$. Since $f \equiv 0 \pmod{h}$ we conclude $i_0(f,[f,g]) = +\infty$ by properties of the intersection multiplicity.

Case 2: $i_0(f,g) = +\infty$. There exists an irreducible power series $h \in K[[x,y]]$ such that $f \equiv 0 \pmod{h}$ and $g \equiv 0 \pmod{h}$. Let $f = a \cdot h$ and $g = b \cdot h$ for some $a, b \in K[[x,y]]$. Observe that $\frac{\partial f}{\partial x}\frac{\partial g}{\partial y} \equiv ab\frac{\partial h}{\partial x}\frac{\partial h}{\partial y} \equiv \frac{\partial f}{\partial y}\frac{\partial g}{\partial x} \pmod{h}$, hence $[f,g] \equiv 0 \pmod{h}$. Since h is an irreducible factor of f and $[f,g]$ we conclude $i_0(f,[f,g]) = +\infty$.

Suppose now that $i_0(f,[f,g]) = +\infty$. There is an irreducible power series $h \in K[[x,y]]$ such that $f \equiv 0 \pmod{h}$ and $[f,g] \equiv 0 \pmod{h}$. If $f \equiv 0 \pmod{h^2}$ then, by the second part of Lemma 3.1, $\mu(f) = +\infty$. Suppose that $f = hf_1$ for some $f_1 \in K[[x,y]]$ with $f_1 \not\equiv 0 \pmod{h}$. We have $[f,g] = [hf_1, g] = h[f_1,g] + f_1[h,g]$. Since h is an irreducible factor of $[f,g]$, we get $f_1\,[h,g] \equiv 0 \pmod{h}$. Let $\gamma(t) := (x(t), y(t))$ be a good parametrization of h. By Property 3.2 we may assume, without loss of generality, that $\frac{\partial h}{\partial y}(\gamma(t)) \neq 0$.

From the identity $h(\gamma(t)) = 0$ we get

$$\frac{\partial h}{\partial x}(\gamma(t))x'(t) + \frac{\partial h}{\partial y}(\gamma(t))y'(t) = 0. \tag{10}$$

On the other hand, since h is an irreducible factor of $[h, g]$, we get $[h, g](\gamma(t)) = 0$, hence

$$\frac{\partial h}{\partial x}(\gamma(t))\frac{\partial g}{\partial y}(\gamma(t)) + \frac{\partial h}{\partial y}(\gamma(t))\left(-\frac{\partial g}{\partial x}(\gamma(t))\right) = 0. \tag{11}$$

From (10) and (11) we get that the pair $\left(\frac{\partial h}{\partial x}(\gamma(t)), \frac{\partial h}{\partial y}(\gamma(t))\right)$ is a nonzero solution of the system, in the unknowns U and V:

$$\begin{cases} x'(t)U + y'(t)V = 0 \\ \frac{\partial g}{\partial y}(\gamma(t))U + \left(-\frac{\partial g}{\partial x}(\gamma(t))\right)V = 0. \end{cases} \tag{12}$$

Hence the determinant of the matrix associated with system (12) equals $\frac{d}{dt}g(\gamma(t)) = 0$ so $g(\gamma(t)) = 0$. Given that h is a common factor of f and g, we conclude that $i_0(f, g) = +\infty$. □

Remark 3.4. *Proposition 3.3 does not hold when the characteristic p of the field K is positive: consider $f(x, y) = y^p + x^{p+1}$ and $g(x, y) = x + y$, then $\mu(f) = +\infty$ but $i_0(f, [f, g]) = p^2$.*

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